

Long-time dynamics of Kirchhoff wave models with strong nonlinear damping

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Abstract

We study well-posedness and long-time dynamics of a class of quasilinear wave equations with a strong damping. We accept the Kirchhoff hypotheses and assume that the stiffness and damping coefficients are C^1 functions of the L_2 -norm of the gradient of the displacement. We first prove the existence and uniqueness of weak solutions and study their properties for a rather wide class of nonlinearities which covers the case of possible degeneration (or even negativity) of the stiffness coefficient and the case of a supercritical source term. Our main results deal with global attractors. In the case of strictly positive stiffness factors we prove that in the natural energy space endowed with a partially strong topology there exists a global attractor whose fractal dimension is finite. In the non-supercritical case the partially strong topology becomes strong and a finite dimensional attractor exists in the strong topology of the energy space. Moreover, in this case we also establish the existence of a fractal exponential attractor and give conditions that guarantee the existence of a finite number of determining functionals. Our arguments involve a recently developed method based on “compensated” compactness and quasi-stability estimates.

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1 Introduction

In a bounded smooth domain $\Omega \subset \mathbb{R}^d$ we consider the following Kirchhoff wave model with a strong nonlinear damping:

$$\begin{cases} \partial_{tt}u - \sigma(\|\nabla u\|^2)\Delta\partial_t u - \phi(\|\nabla u\|^2)\Delta u + f(u) = h(x), & x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \quad (1)$$

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Here Δ is the Laplace operator, σ and ϕ are scalar functions specified later, $f(u)$ is a given source term, h is a given function in $L^2(\Omega)$ and $\|\cdot\|$ is the norm in $L^2(\Omega)$.

This kind of wave models goes back to G. Kirchhoff ($d = 1$, $\phi(s) = \varphi_0 + \varphi_1 s$, $\sigma(s) \equiv 0$, $f(u) \equiv 0$) and has been studied by many authors under different types of hypotheses. We refer to [4, 28, 41] and to the literature cited in the survey [32], see also [5, 17, 19, 22, 31, 34, 35, 36, 37, 46, 47, 48, 49] and the references therein.

Our main goal in this paper is to study well-posedness and long-time dynamics of the problem (1) under the following set of hypotheses:

Assumption 1.1 (i) The damping (σ) and the stiffness (ϕ) factors are C^1 functions on the semi-axis $\mathbb{R}_+ = [0, +\infty)$. Moreover, $\sigma(s) > 0$ for all $s \in \mathbb{R}_+$ and there exist $c_i \geq 0$ and $\eta_0 \geq 0$ such that

$$\int_0^s [\phi(\xi) + \eta_0 \sigma(\xi)] d\xi \rightarrow +\infty \quad \text{as } s \rightarrow +\infty \quad (2)$$

and

$$s\phi(s) + c_1 \int_0^s \sigma(\xi) d\xi \geq -c_2 \quad \text{for } s \in \mathbb{R}_+. \quad (3)$$

(ii) $f(u)$ is a C^1 function such that $f(0) = 0$ (without loss of generality),

$$\mu_f := \liminf_{|s| \rightarrow \infty} \{s^{-1} f(s)\} > -\infty, \quad (4)$$

and the following properties hold: (a) if $d = 1$, then f is arbitrary; (b) if $d = 2$ then

$$|f'(u)| \leq C(1 + |u|^{p-1}) \quad \text{for some } p \geq 1;$$

(c) if $d \geq 3$ then either

$$|f'(u)| \leq C(1 + |u|^{p-1}) \quad \text{with some } 1 \leq p \leq p_* \equiv \frac{d+2}{d-2}, \quad (5)$$

or else

$$c_0 |u|^{p-1} - c_1 \leq f'(u) \leq c_2 (1 + |u|^{p-1}) \quad \text{with some } p_* < p < p_{**} \equiv \frac{d+4}{(d-4)_+}, \quad (6)$$

where c_i are positive constants and $s_+ = (s + |s|)/2$.

Remark 1.2 (1) The coercive behavior in (2) and (3) holds with $\eta_0 = c_1 = 0$ if we assume that $\liminf_{s \rightarrow +\infty} \{s\phi(s)\} > 0$, for instance. The standard example is $\phi(s) = \phi_0 + \phi_1 s^\alpha$ with $\phi_0 \in \mathbb{R}$, $\phi_1 > 0$ and $\alpha \geq 1$. However we can also take $\phi(s)$ with finite support, or even $\phi(s) \equiv \text{const} \leq 0$. In this case we need additional hypotheses concerning behavior of $\sigma(s)$ as $s \rightarrow +\infty$. We note that the physically justified situation (see, e.g., the survey [32]) corresponds to the case when the stiffness coefficient $\phi(s)$ is positive almost everywhere. However we include into the consideration the case of possibly negative ϕ because the argument we use to prove well-posedness involves positivity properties of ϕ in a rather mild form (see, e.g., (2) and (3)).

(2) We note that in the case when $d \leq 2$ or $d \geq 3$ and (5) holds with $p < p_*$ the Nemytski operator $u \mapsto f(u)$ is a locally Lipschitz mapping from the Sobolev space $H_0^1(\Omega)$ into $H^{-1+\delta}(\Omega)$

for some $\delta > 0$. If $d \geq 3$ and (5) holds with $p = p_*$ this fact is valid with $\delta = 0$. These properties of the source nonlinearity $f(u)$ are of importance in the study of wave dynamics with the strong damping (see, e.g., [6, 7, 38, 45] and the references therein). Below we refer to this situation as to non-supercritical (subcritical when $\delta > 0$ and critical for the case $\delta = 0$). To deal with the supercritical case (the inequality in (5) holds with $p > p_*$) we borrow some ideas from [23] and we need a lower bound for $f(u)$ of the same order as its upper bound (see the requirement in (6)). The second critical exponent p_{**} arises in the dimension $d \geq 5$ from the requirement $H^2(\Omega) \subset L_{p+1}(\Omega)$ which we need to estimate the source term in some negative Sobolev space, see also Remark 2.6 below.

(3) We also note that in the case (6) the condition in (4) holds automatically (with $\mu_f = +\infty$). This condition can be relaxed depending on the properties of ϕ . For instance, in the case when $\phi(s) = \phi_0 + \phi_1 s^\alpha$ with $\phi_1 > 0$ instead of (4) we can assume that

$$f(s)s \geq -c_1 |s|^l - c_2 \quad \text{for some } l \leq \min\{2\alpha + 2 - \varepsilon, 2d/(d-2)_+\}$$

with arbitrary small $\varepsilon > 0$. Therefore for this choice of ϕ we need no coercivity assumptions concerning f in the non-supercritical case provided $p < 2\alpha + 1$. However we do not pursue these possible generalizations and prefer to keep hypotheses concerning ϕ and σ as general as possible.

Well-posedness issues for Kirchhoff type models like (1) were studied intensively last years. The main attention was paid the case when the strong damping term $-\sigma \Delta u_t$ is absent and the source term $f(u)$ is either absent or subcritical. We refer to [19, 36, 49] and also to the survey [32]. In these papers the authors have studied sets of initial data for which solutions exist and are unique. The papers [19, 36] consider also the case of a degenerate stiffness coefficient ($\phi(s) \sim s^\alpha$ near zero). We also mention the paper [31] which deals with global existence (for a restricted class of initial data) in the case of a strictly positive stiffness factor of the form $\phi(s) = \phi_0 + \phi_1 s^\alpha$ with the nonlinear damping $|u_t|^q u_t$ and the source term $f(u) = -|u|^p u$ for some range of exponents q and p , see also the recent paper [43] which is concentrated on the *local* existence issue for the same type of damping and source terms but for a wider range of the exponents p and q .

Introducing of the strong (Kelvin-Voigt) damping term $-\sigma \Delta u_t$ provides an additional a priori estimate and simplifies the issue. There are several well-posedness results available in the literature for this case (see [5, 33, 35, 37, 46, 48, 47]). However all these publications assume that the damping coefficient $\sigma(s) \equiv \sigma_0 > 0$ is a constant and deal with a subcritical or absent source term. Moreover, all of them (except [37]) assume that stiffness factor is non-degenerate (i.e., $\phi(s) \geq \phi_0 > 0$). However [37] assumes small initial energy, i.e., deals with local (in phase space) dynamics. Recently the existence and uniqueness of weak (energy) solutions of (1) was reported (without detailed proofs) in [23] for the case of supercritical source satisfying (6). However the authors in [23] assume (in addition to our hypotheses) that $d = 3$, the damping is linear (i.e., $\sigma(s) = \text{const} > 0$) and the stiffness factor ϕ is a uniformly positive C^1 function satisfying the inequality $\int_0^s \phi(\xi) d\xi \leq s\phi(s)$ for all $s \geq 0$. As for *nonlinear strong* damping to the best of our knowledge there is only one publication [26]. This paper deals with nonlinear damping of the form $\sigma(\|A^\alpha u\|^2) A^\alpha u_t$ with $0 < \alpha \leq 1$. The main result of [26] states only the existence of weak solutions for uniformly positive ϕ and σ in the case when $f(u) \equiv 0$.

The main achievement of our well-posedness result is that (a) we do not assume any kind of non-degeneracy conditions concerning ϕ (this function may be zero or even negative); (b) we

consider a nonlinear state-dependent strong damping and do not assume uniform positivity of the damping factor σ ; (c) we cover the cases of critical and supercritical source terms f .

Our second result deals with a global attractor for the dynamical system generated by (1). There are many papers on stabilization to zero equilibrium for Kirchhoff type models (see, e.g., [1, 5, 31, 32] and the references therein) and only a few recent results devoted to (non-trivial) attractors for systems like (1). We refer to [34] for studies of local attractors in the case of viscous damping and to [17, 35, 46, 47, 48] in the case of a strong *linear* damping (possibly perturbed by nonlinear viscous terms). All these papers assume subcriticality of the force $f(u)$ and deal with a uniformly positive stiffness coefficient of the form $\phi(s) = \phi_0 + \phi_1 s^\alpha$ with $\phi_0 > 0$. In the long time dynamics context we can point only the paper [1] which contains a result (see Theorem 4.4[1]) on stabilization to zero in the case when $\phi(s) \equiv \sigma(s) = a + bs^\gamma$ with $a > 0$ and possibly supercritical source with the property $f(u)u + a\mu u^2 \geq 0$, where $\mu > 0$ is small enough. In this case the global attractor $\mathfrak{A} = \{0\}$ is trivial. However this paper does not discuss well-posedness issues and assumes the existence of sufficiently smooth solutions as a starting point of the whole considerations.

Our main novelty is that we consider long-time dynamics for much more general stiffness and damping coefficients and cover the supercritical case. Namely, under some additional non-degeneracy assumptions we prove the existence of a finite dimensional global attractor which uniformly attracts trajectories in a partially strong sense (see Definition 3.1). In the non-supercritical case this result can be improved: we establish the convergence property with respect to strong topology of the phase (energy) space. Moreover, in this case we prove the existence of a fractal exponential attractor and give conditions for the existence of finite sets of determining functionals. To establish these results we rely on recently developed approach (see [12] and also [13] and [14, Chapters 7,8]) which involves stabilizability estimates, the notion of a quasi-stable system and also the idea of "short" trajectories due to [29, 30]. In the supercritical case to prove that the attractor has a finite dimension we also use a recent observation made in [23] concerning stabilizability estimate in the extended space. In the non-supercritical case we first prove that the corresponding system is quasi-stable in the sense of the definition given in [14, Section 7.9] and then apply the general theorems on properties of quasi-stable systems from this source.

We also note that long-time dynamics of second order equations with nonlinear damping was studied by many authors. We refer to [3, 11, 20, 24, 39, 40] for the case of a damping with a displacement-dependent coefficient and to [12, 13, 14] and to the references therein for a velocity-dependent damping. Models with different types of strong (linear) damping in wave equations were considered in [6, 7, 23, 38, 45], see also the literature quoted in these references.

The paper is organized as follows. In Section 2 we introduce some notations and prove Theorem 2.2 which provides us with well-posedness of our model and contains some additional properties of solutions. In Section 3 we study long-time dynamics of the evolution semigroup $S(t)$ generated by (1). We first establish some continuity properties of $S(t)$ (see Proposition 3.2) and its dissipativity (Proposition 3.5). These results do not require any non-degeneracy hypotheses concerning the stiffness coefficient ϕ . Then in the case of strictly positive ϕ we prove asymptotic compactness of $S(t)$ (see Theorem 3.9 and Corollary 3.10). Our main results in Section 3 state the existence of global attractors and describe their properties in both the general case (Theorems 3.11 and 3.13) and the non-supercritical case (Theorems 3.16 and 3.18).

2 Well-posedness

We first describe some notations.

Let $H^\sigma(\Omega)$ be the L_2 -based Sobolev space of the order σ with the norm denoted by $\|\cdot\|_\sigma$ and $H_0^\sigma(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^\sigma(\Omega)$ for $\sigma > 0$. Below we also denote by $\|\cdot\|$ and (\cdot, \cdot) the norm and the inner product in $L_2(\Omega)$.

In the space $H = L_2(\Omega)$ we introduce the operator $\mathcal{A} = -\Delta_D$ with the domain

$$\mathcal{D}(\mathcal{A}) = \{u \in H^2(\Omega) : u = 0 \text{ on } \Omega\} \equiv H^2(\Omega) \cap H_0^1(\Omega),$$

where Δ_D is the Laplace operator in Ω with the Dirichlet boundary conditions. The operator \mathcal{A} is a linear self-adjoint positive operator densely defined on $H = L_2(\Omega)$. The resolvent of \mathcal{A} is compact in H . Below we denote by $\{e_k\}$ the orthonormal basis in H consisting of eigenfunctions of the operator \mathcal{A} :

$$\mathcal{A}e_k = \lambda_k e_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{k \rightarrow \infty} \lambda_k = \infty.$$

We also denote $\mathcal{H} = [H_0^1(\Omega) \cap L_{p+1}(\Omega)] \times L_2(\Omega)$. In the non-supercritical case (when $d \leq 2$ or $d \geq 3$ and $p \leq p_* = (d-2)(d+2)^{-1}$) we have that $H_0^1(\Omega) \subset L_{p+1}(\Omega)$ ¹ and thus the space \mathcal{H} coincides with $H_0^1(\Omega) \times L_2(\Omega)$. We define the norm in \mathcal{H} by the relation

$$\|(u_0; u_1)\|_{\mathcal{H}}^2 = \|\nabla u_0\|^2 + \alpha \|u_0\|_{L_{p+1}(\Omega)}^2 + \|u_1\|^2, \quad (7)$$

where $\alpha = 1$ in the case when $d \geq 3$ and $p > p_*$ and $\alpha = 0$ in other cases.

Definition 2.1 A function $u(t)$ is said to be a weak solution to (1) on an interval $[0, T]$ if

$$u \in L_\infty(0, T; H_0^1(\Omega) \cap L_{p+1}(\Omega)), \quad \partial_t u \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)) \quad (8)$$

and (1) is satisfied in the sense of distributions.

Our main result in this section is Theorem 2.2 on well-posedness of problem (1). This theorem also contains some auxiliary properties of solutions which we need for the results on the asymptotic dynamics.

Theorem 2.2 (Well-posedness) *Let Assumption 1.1 be in force and $(u_0; u_1) \in \mathcal{H}$. Then for every $T > 0$ problem (1) has a unique weak solution $u(t)$ on $[0, T]$. This solution possesses the following properties:*

1. *The function $t \mapsto (u(t); u_t(t))$ is (strongly) continuous in $\mathcal{H} = [H_0^1 \cap L_{p+1}](\Omega) \times L_2(\Omega)$ and*

$$u_{tt} \in L_2(0, T; H^{-1}(\Omega)) + L_\infty(0, T; L_{1+1/p}(\Omega)). \quad (9)$$

Moreover, there exists a constant $C_{R,T} > 0$ such that

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + c_0 \|u(t)\|_{L_{p+1}(\Omega)}^2 + \int_0^t \|\nabla u_t(\tau)\|^2 d\tau \leq C_{R,T} \quad (10)$$

¹To unify the presentation we suppose that $p \geq 1$ is arbitrary in all appearances in the case $d = 1$.

for every $t \in [0, T]$ and initial data $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$, where $c_0 = 1$ in the case when (6) holds and $c_0 = 0$ in other cases. We also have the following additional regularity:

$$u_t \in L_{\infty}(a, T; H_0^1(\Omega)), \quad u_{tt} \in L_{\infty}(a, T; H^{-1}(\Omega)) \cap L_2(a, T; L_2(\Omega))$$

for every $0 < a < T$ and there exist $\beta > 0$ and $c_{R,T} > 0$ such that

$$\|u_{tt}(t)\|_{-1}^2 + \|\nabla u_t(t)\|^2 + \int_t^{t+1} \left[\|u_{tt}(\tau)\|^2 + c_0 \int_{\Omega} |u(x, \tau)|^{p-1} |u_t(x, \tau)|^2 dx \right] d\tau \leq \frac{c_{R,T}}{t^{\beta}} \quad (11)$$

for every $t \in (0, T]$, where as above $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$ and $c_0 > 0$ in the supercritical case only.

2. The following energy identity

$$\mathcal{E}(u(t), u_t(t)) + \int_s^t \sigma(\|\nabla u(\tau)\|^2) \|\nabla u_t(\tau)\|^2 d\tau = \mathcal{E}(u(s), u_t(s)) \quad (12)$$

holds for every $t > s \geq 0$, where the energy \mathcal{E} is defined by the relation

$$\mathcal{E}(u_0, u_1) = \frac{1}{2} [\|u_1\|^2 + \Phi(\|\nabla u_0\|^2)] + \int_{\Omega} F(u_0) dx - \int_{\Omega} h u_0 dx, \quad (u_0; u_1) \in \mathcal{H},$$

with

$$\Phi(s) = \int_0^s \phi(\xi) d\xi \quad \text{and} \quad F(s) = \int_0^s f(\xi) d\xi.$$

3. If $u^1(t)$ and $u^2(t)$ are two weak solutions such that $\|(u^i(0); u_t^i(0))\|_{\mathcal{H}} \leq R$, $i = 1, 2$, then there exists $b_{R,T} > 0$ such that the difference $z(t) = u^1(t) - u^2(t)$ satisfies the relation

$$\|z_t(t)\|_{-1}^2 + \|\nabla z(t)\|^2 + \int_0^t \|z_t(\tau)\|^2 d\tau \leq b_{R,T} (\|z_t(0)\|_{-1}^2 + \|\nabla z(0)\|^2) \quad (13)$$

for all $t \in [0, T]$, and, if (6) holds, we also have that

$$\int_0^T \left[\int_{\Omega} |z|^{p+1} dx + \int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \right] d\tau \leq b_{R,T} (\|z_t(0)\|_{-1}^2 + \|\nabla z(0)\|^2). \quad (14)$$

4. If we assume in addition that $u_0 \in (H^2 \cap H_0^1)(\Omega)$, then $u \in C_w(0, T; (H^2 \cap H_0^1)(\Omega))$, where $C_w(0, T; X)$ stands for the space of weakly continuous functions with values in X , and under the condition $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$ we have that

$$\|u_t(t)\|^2 + \|\Delta u(t)\|^2 \leq C_R(T) (1 + \|\Delta u_0\|^2) \quad \text{for every } t \in [0, T]. \quad (15)$$

Proof. Let $\Sigma(s) = \int_0^s \sigma(\xi) d\xi$. For every $\eta > 0$ we introduce the following functional on \mathcal{H} :

$$\mathcal{E}_+^{\eta}(u_0, u_1) = \|u_1\|^2 + [\Phi(\|\nabla u_0\|^2) + \eta \Sigma(\|\nabla u_0\|^2) - a(\eta)] + \alpha \|u_0\|_{L_{p+1}^{p+1}(\Omega)}^{p+1} + \|u_0\|^2 \quad (16)$$

with $a(\eta) = \inf_{s \in \mathbb{R}_+} \{\Phi(s) + \eta \Sigma(s)\}$, where $\alpha = 1$ in the case when (6) holds and $\alpha = 0$ in other cases. By (2) this functional is finite for every $\eta \geq \eta_0$.

Let $\nu \in \mathbb{R}_+$ and

$$\mathcal{W}^{\eta,\nu}(u_0, u_1) = \mathcal{E}(u_0, u_1) + \eta \left[(u_0, u_1) + \frac{1}{2} \Sigma(\|\nabla u_0\|^2) \right] + \nu \|u_0\|^2. \quad (17)$$

One can see that for every $\eta \geq \eta_0$ we can choose $\nu = \nu(\eta, \mu_f) \geq 0$, positive constants a_i and a monotone positive function $M(s)$ such that

$$a_0 \mathcal{E}_+^\eta(u_0, u_1) - a_1 \leq \mathcal{W}^{\eta,\nu}(u_0, u_1) \leq a_2 \mathcal{E}_+^\eta(u_0, u_1) + M(\|\nabla u_0\|^2), \quad \forall (u_0; u_1) \in \mathcal{H}. \quad (18)$$

To prove the existence of solutions, we use the standard Galerkin method. We start with the case when $u_0 \in (H^2 \cap H_0^1)(\Omega)$ and assume that $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$ for some $R > 0$. We seek for approximate solutions of the form

$$u^N(t) = \sum_{k=1}^N g_k(t) e_k, \quad N = 1, 2, \dots,$$

that satisfy the finite-dimensional projections of (1). Moreover, we assume that

$$\|(u^N(0); u_t^N(0))\|_{\mathcal{H}} \leq C_R \quad \text{and} \quad \|u^N(0) - u_0\|_2 \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Such solutions exist (at least locally), and after multiplication of the corresponding projection of (1) by $u_t^N(t)$ we get that $u^N(t)$ satisfies the energy relation in (12). Similarly, one can see from (3) and (4) that

$$\begin{aligned} \frac{d}{dt} \left[(u^N, u_t^N) + \frac{1}{2} \Sigma(\|\nabla u^N\|^2) \right] &= \|u_t^N\|^2 - \phi(\|\nabla u^N\|^2) \|\nabla u^N\|^2 - (f(u^N), u^N) + (h, u^N) \\ &\leq \|u_t^N\|^2 + C_1 \Sigma(\|\nabla u^N\|^2) + C_2 \|u^N\|^2 + C_3. \end{aligned}$$

One can see from (2) that for every $\eta > \eta_0$ there exist $c_i > 0$ such that

$$\Sigma(s) \leq c_1 [\Phi(s) + \eta \Sigma(s) - a(\eta)] + c_2, \quad s \in \mathbb{R}_+.$$

Thus using (18) we have that the function $\mathcal{W}_N^{\eta,\nu}(t) \equiv \mathcal{W}^{\eta,\nu}(u^N(t), u_t^N(t))$ satisfies the inequality

$$\frac{d}{dt} \mathcal{W}_N^{\eta,\nu}(t) \leq \eta (\|u_t^N\|^2 + C_1 \Sigma(\|\nabla u^N\|^2) + C_2 \|u^N\|^2 + C_3) \leq c_1 \mathcal{W}_N^{\eta,\nu}(t) + c_2$$

for $\eta > \eta_0$ with ν depending on η and f . Therefore, using Gronwall's type argument and also relation (18) we obtain

$$\mathcal{E}_+^\eta(u^N(t); u_t^N(t)) \leq C_{R,T} \quad \text{for all } t \in [0, T], \quad N = 1, 2, 3, \dots,$$

for every $\eta > \eta_0$. By the coercivity requirement in (2) we conclude that

$$\|(u^N(t); u_t^N(t))\|_{\mathcal{H}} \leq C_{R,T} \quad \text{for all } t \in [0, T], \quad N = 1, 2, 3, \dots \quad (19)$$

Since $\sigma(s) > 0$, this implies that $\sigma(\|\nabla u^N(t)\|^2) > \sigma_{R,T}$ for all $t \in [0, T]$. Therefore the energy relation (12) for u^N yields that

$$\int_0^T \|\nabla u_t^N(t)\|^2 dt \leq C(R, T), \quad N = 1, 2, \dots, \quad \text{for any } T > 0. \quad (20)$$

Now we use the multiplier $-\Delta u$ (below we omit the superscript N for shortness). We obviously have that

$$\begin{aligned} \frac{d}{dt} \left[-(u_t, \Delta u) + \frac{1}{2} \sigma(\|\nabla u\|^2) \|\Delta u\|^2 \right] + \phi(\|\nabla u\|^2) \|\Delta u\|^2 + (f'(u), |\nabla u|^2) \\ \leq \|\nabla u_t\|^2 + \sigma'(\|\nabla u\|^2) (\nabla u, \nabla u_t) \|\Delta u\|^2 + \|h\| \|\Delta u\|. \end{aligned} \quad (21)$$

In the case when $d \geq 3$ and (6) holds, we have

$$(f'(u), |\nabla u|^2) \geq c_0 \int_{\Omega} |u|^{p-1} |\nabla u|^2 dx - c_1 \|\nabla u\|^2, \quad c_0, c_1 > 0.$$

In other (non-supercritical) cases, due to the embedding $H^1(\Omega) \subset L_{p+1}(\Omega)$, from (19) we have the relation $|(f'(u), |\nabla u|^2)| \leq c_{R,T} \|\Delta u\|^2$. This implies that

$$\frac{d}{dt} \left[-(u_t, \Delta u) + \frac{1}{2} \sigma(\|\nabla u\|^2) \|\Delta u\|^2 \right] \leq \|\nabla u_t\|^2 + c_{R,T} (1 + \|\nabla u_t\|) \cdot \|\Delta u\|^2 + C_{R,T}. \quad (22)$$

for every $t \in [0, T]$. Let

$$\Psi(t) = \mathcal{E}(u(t), u_t(t)) + \eta \left[-(u_t, \Delta u) + \frac{1}{2} \sigma(\|\nabla u\|^2) \|\Delta u\|^2 \right]$$

with $\eta > 0$. We note that there exists $\eta_* = \eta(R, T) > 0$ such that

$$\Psi(t) \geq \alpha_{R,T,\eta} [\|u_t\|^2 + \|\Delta u\|^2] - C_{R,T}, \quad t \in [0, T], \quad (23)$$

for every $0 < \eta < \eta_*$. Therefore using the energy relation (12) for the approximate solutions and also (22) one can choose $\eta > 0$ such that

$$\frac{d}{dt} \Psi(t) \leq c_0 [\Psi(t) + c_1] (1 + \|\nabla u_t\|^2), \quad t \in [0, T],$$

with appropriate $c_i > 0$. By (20) and (23) this implies the estimate

$$\|u_t^N(t)\|^2 + \|\Delta u^N(t)\|^2 \leq C_R(T) [1 + \|\Delta u^N(0)\|^2], \quad t \in [0, T].$$

The above a priori estimates show that $(u_N; \partial_t u_N)$ is *-weakly compact in

$$\mathcal{W}_T \equiv L_{\infty}(0, T; H^2(\Omega)) \cap L_{p+1}(\Omega) \times [L_{\infty}(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega))] \quad \text{for every } T > 0.$$

Moreover, using the equation for $u^N(t)$ we can show in the standard way that

$$\int_0^T \|\partial_{tt} u^N(t)\|_{-m}^2 dt \leq C_T(R), \quad N = 1, 2, \dots, \quad (24)$$

for some $m \geq \max\{1, d/2\}$. Thus the Aubin-Dubinsky theorem (see [42, Corollary 4]) yields that $(u_N; \partial_t u_N)$ is also compact in

$$C(0, T; H^{2-\varepsilon}(\Omega)) \times [C(0, T; H^{-\varepsilon}(\Omega)) \cap L_2(0, T; H^{1-\varepsilon}(\Omega))] \quad \text{for every } \varepsilon > 0.$$

Thus there exists an element $(u; u_t)$ in \mathcal{W}_T such that (along a subsequence) the following convergence holds:

$$\max_{[0,T]} \|u^N(t) - u(t)\|_{2-\varepsilon}^2 + \int_0^T \|u_t^N(t) - u_t(t)\|_{1-\varepsilon}^2 dt \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Moreover, by the Lions Lemma (see Lemma 1.3 in [27, Chap.1]) we have that

$$f(u^N(x, t)) \rightarrow f(u(x, t)) \quad \text{weakly in } L_{1+1/p}([0, T] \times \Omega).$$

This allows us to make a limit transition in nonlinear terms and prove the existence of a weak solution under the additional condition $u_0 \in (H^2 \cap H_0^1)(\Omega)$. One can see that this solution possesses the properties (9), (10), (15) and satisfies the corresponding energy inequality.

Now we prove that (13) (and also (14) in the supercritical case) hold for every couple $u^1(t)$ and $u^2(t)$ of weak solutions. For this we use the same idea as [23] and start with the following preparatory lemma which we also use in the further considerations.

Lemma 2.3 *Let $u^1(t)$ and $u^2(t)$ be two weak solutions to (1) with different initial data $(u_0^i; u_1^i)$ from \mathcal{H} such that*

$$\|u_t^i(t)\|^2 + \|\nabla u^i(t)\|^2 \leq R^2 \quad \text{for all } t \in [0, T] \quad \text{and for some } R > 0. \quad (25)$$

Then for $z(t) = u^1(t) - u^2(t)$ we have the relation

$$\begin{aligned} \frac{d}{dt} \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \cdot \|\nabla z\|^2 \right] + \frac{1}{2} \phi_{12}(t) \cdot \|\nabla z\|^2 + (f(u^1) - f(u^2), z) \\ + \tilde{\phi}_{12}(t) |(\nabla(u^1 + u^2), \nabla z)|^2 \leq \|z_t\|^2 + C_R (\|\nabla u_t^1\| + \|\nabla u_t^2\|) \|\nabla z\|^2 \end{aligned} \quad (26)$$

for all $t \in [0, T]$, where $\sigma_{12}(t) = \sigma_1(t) + \sigma_2(t)$ and $\phi_{12}(t) = \phi_1(t) + \phi_2(t)$ with $\sigma_i(t) = \sigma(\|\nabla u^i(t)\|^2)$ and $\phi_i(t) = \phi(\|\nabla u^i(t)\|^2)$. We also use the following notation

$$\tilde{\phi}_{12}(t) = \frac{1}{2} \int_0^1 \phi'(\lambda \|\nabla u^1(t)\|^2 + (1 - \lambda) \|\nabla u^2(t)\|^2) d\lambda. \quad (27)$$

Remark 2.4 It follows directly from Definition 2.1 that (9) holds for every weak solution. This and also (8) allows us to show that $(z, z_t) + \sigma_{12}(t) \|\nabla z\|^2 / 4$ is absolutely continuous with respect to t and thus the relation in (26) has a meaning for every couple of weak solutions.

Proof. One can see that $z(t) = u^1(t) - u^2(t)$ solves the equation

$$z_{tt} - \frac{1}{2} \sigma_{12}(t) \Delta z_t - \frac{1}{2} \phi_{12}(t) \Delta z + G(u^1, u^2; t) = 0, \quad (28)$$

where

$$G(u^1, u^2; t) = -\frac{1}{2} \{ [\sigma_1(t) - \sigma_2(t)] \Delta(u_t^1 + u_t^2) + [\phi_1(t) - \phi_2(t)] \Delta(u^1 + u^2) \} + f(u^1) - f(u^2).$$

Since $G \in L_2(0, T; H^{-1}(\Omega)) + L_\infty(0, T; L_{1+1/p}(\Omega))$ and $z \in L_\infty(0, T; (H_0^1 \cap L_{p+1})(\Omega))$ for any couple u^1 and u^2 of weak solutions, we can multiply equation (28) by z in $L_2(\Omega)$. Therefore using the relation

$$|\sigma'_{12}(t)| \leq C_R (\|\nabla u_t^1\| + \|\nabla u_t^2\|)$$

and also the observation made in Remark 2.4 we conclude that

$$\begin{aligned} \frac{d}{dt} \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \cdot \|\nabla z\|^2 \right] + \frac{1}{2} \phi_{12}(t) \cdot \|\nabla z\|^2 + (G(u^1, u^2, t), z) \\ \leq \|z_t\|^2 + C_R (\|\nabla u_t^1\| + \|\nabla u_t^2\|) \cdot \|\nabla z\|^2. \end{aligned}$$

One can see that $\phi_1(t) - \phi_2(t) = 2(\nabla(u^1 + u^2), \nabla z) \cdot \tilde{\phi}_{12}(t)$, where $\tilde{\phi}_{12}$ is given by (27), and

$$|[\sigma_1(t) - \sigma_2(t)](\nabla(u_t^1 + u_t^2), \nabla z)| \leq C_R (\|\nabla u_t^1\| + \|\nabla u_t^2\|) \cdot \|\nabla z\|^2.$$

Thus using the structure of the term $G(u^1, u^2; t)$ we obtain (26). \square

Lemma 2.5 *Assume that $f(u)$ satisfies Assumption 1.1 and the additional requirement² saying that $f'(u) \geq -c$ for some $c \geq 0$. Then for $z = u^1 - u^2$ we have that*

$$\int_{\Omega} (f(u^1) - f(u^2))(u^1 - u^2) dx \geq -c_0 \|z\|^2 + c_1 \int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \quad (29)$$

and

$$\int_{\Omega} (f(u^1) - f(u^2))(u^1 - u^2) dx \geq -c_0 \|z\|^2 + c_1 \int_{\Omega} |z|^{p+1} dx, \quad (30)$$

where $c_0 \geq 0$ and $c_1 > 0$ in the case when (6) holds and $c_1 = 0$ in other cases.

Proof. It is sufficient to consider the case when (6) holds.

The relation in (29) follows from the obvious inequality

$$\int_0^1 |(1-\lambda)u^1 + \lambda u^2|^r d\lambda \geq c_r (|u^1|^r + |u^2|^r), \quad r \geq 0, \quad u^i \in \mathbb{R},$$

which can be obtained by the direct calculation of the integral. As for (30) we use the obvious representation

$$\int_{u^1}^{u^2} |\xi|^r d\xi = \frac{1}{r+1} (|u^1|^r u^1 - |u^2|^r u^2), \quad r \geq 0, \quad u^i \in \mathbb{R}, \quad u^1 < u^2,$$

and the argument given in [14, Remark 3.2.9]. \square

Now we return to the proof of relations (13) and (14).

Let u^1 and u^2 be weak solutions satisfying (25) and also the inequality $\|u^i(t)\|_{L_{p+1}(\Omega)} \leq R$ for all $t \in [0, T]$ in the supercritical case. We first note that in the non-supercritical case by the

²This requirement holds automatically in the supercritical case, see (6).

embedding $H^1(\Omega) \subset L_r(\Omega)$ for $r = \infty$ in the case $d = 1$, for arbitrary $1 \leq r < \infty$ when $d = 2$ and for $r = 2d(d-2)^{-1}$ in the case $d \geq 3$ we have that

$$\|f(u^1) - f(u^2)\|_{-1} \leq C_R \|\nabla(u^1 - u^2)\|, \quad u^1, u^2 \in H_0^1(\Omega), \quad \|\nabla u^i\| \leq R, \quad (31)$$

which implies that $|(f(u^1) - f(u^2), z)| \leq C_R \|\nabla z\|^2$. Therefore it follows from Lemma 2.3 and from Lemma 2.5 in the supercritical case that

$$\begin{aligned} \frac{d}{dt} \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \|\nabla z\|^2 \right] + \frac{1}{2} \phi_{12}(t) \|\nabla z\|^2 + c_0 \left[\int_{\Omega} |z|^{p+1} dx + \int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \right] \\ \leq \|z_t\|^2 + C_R (1 + \|\nabla u_t^1\| + \|\nabla u_t^2\|) \|\nabla z\|^2, \end{aligned} \quad (32)$$

where c_0 is positive in the supercritical case only.

Now we consider the multiplier $\mathcal{A}^{-1} z_t$. Since $H^{2-\eta}(\Omega) \subset L_{p+1}(\Omega)$ for some $\eta > 0$ under the condition $p < p_{**} = (d+4)/(d-4)_+$, we easily obtain that

$$\|\mathcal{A}^{-1} z_t\|_{L_{p+1}}^2 \leq C \|\mathcal{A}^{-\eta/2} z_t\|^2 \leq \varepsilon \|z_t\|^2 + C_\varepsilon \|\mathcal{A}^{-1/2} z_t\|^2 \quad \text{for every } \varepsilon > 0. \quad (33)$$

Thus we can multiply equation (28) by $\mathcal{A}^{-1} z_t$ and obtain that

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{A}^{-1/2} z_t\|^2 + \frac{1}{2} \phi_{12}(t) (z, z_t) + \frac{1}{2} \sigma_{12}(t) \|z_t\|^2 + (G(u^1, u^2; t), \mathcal{A}^{-1} z_t) = 0, \quad (34)$$

where

$$(G(u^1, u^2; t), \mathcal{A}^{-1} z_t) = G_1(t) + G_2(t) + G_3(t). \quad (35)$$

Here

$$G_1(t) = -\frac{1}{2} [\sigma_1(t) - \sigma_2(t)] (\Delta(u_t^1 + u_t^2), \mathcal{A}^{-1} z_t),$$

$$G_2(t) = \tilde{\phi}_{12}(t) (\nabla(u^1 + u^2), \nabla z) (\nabla(u^1 + u^2), \nabla \mathcal{A}^{-1} z_t)$$

with $\tilde{\phi}_{12}(t)$ given by (27), and $G_3(t) = (f(u^1) - f(u^2), \mathcal{A}^{-1} z_t)$.

One can see that $|(G_1(t) + G_2(t))| \leq C_R \|z_t\| \cdot \|\nabla z\|$. In the non-supercritical case by (31) we have the same estimate for $|G_3(t)|$. In the supercritical case we obviously have that

$$\begin{aligned} \int_{\Omega} |f(u^1) - f(u^2)| |\mathcal{A}^{-1} z_t| dx \\ \leq \varepsilon \int_{\Omega} (1 + |u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx + C_\varepsilon \int_{\Omega} (1 + |u^1|^{p-1} + |u^2|^{p-1}) |\mathcal{A}^{-1} z_t|^2 dx \\ \leq \varepsilon \int_{\Omega} (1 + |u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx + C_\varepsilon \left[\int_{\Omega} (1 + |u^1|^{p+1} + |u^2|^{p+1}) dx \right]^{\frac{p-1}{p+1}} \|\mathcal{A}^{-1} z_t\|_{L_{p+1}}^2. \end{aligned} \quad (36)$$

Therefore using (33) we have that

$$\begin{aligned} |(G(u^1, u^2; t), \mathcal{A}^{-1} z_t)| \leq C_R \|z_t\| \cdot \|\nabla z\| \\ + \varepsilon \left[\int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx + \|z\|^2 + \|z_t\|^2 \right] + C_\varepsilon(R) \|\mathcal{A}^{-1/2} z_t\|^2 \end{aligned}$$

for any $\varepsilon > 0$. Thus from (34) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathcal{A}^{-1/2} z_t\|^2 + \frac{1}{2} \sigma_{12}(t) \|z_t\|^2 &\leq C_R \|z_t\| \cdot \|\nabla z\| \\ &+ \varepsilon c_0 \left[\int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx + \|z\|^2 + \|z_t\|^2 \right] + c_0 C_{\varepsilon}(R) \|\mathcal{A}^{-1/2} z_t\|^2 \end{aligned} \quad (37)$$

for any $\varepsilon > 0$, where $c_0 = 0$ in the non-supercritical case. Let

$$\Psi(t) = \frac{1}{2} \|\mathcal{A}^{-1/2} z_t\|^2 + \eta \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \|\nabla z\|^2 \right] \quad (38)$$

for $\eta > 0$ small enough. It is obvious that for $\eta \leq \eta_0(R)$ we have

$$a_R \eta \left[\|\mathcal{A}^{-1/2} z_t\|^2 + \|\nabla z\|^2 \right] \leq \Psi(t) \leq b_R \left[\|\mathcal{A}^{-1/2} z_t\|^2 + \|\nabla z\|^2 \right]. \quad (39)$$

From (32) and (37) we also have that

$$\begin{aligned} \frac{d\Psi}{dt} + \left[\frac{1}{2} \sigma_{12}(t) - \eta - c\varepsilon \right] \|z_t\|^2 + c_0 \eta \int_{\Omega} |z|^{p+1} dx \\ + c_0 (\eta - \varepsilon) \int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \leq C_{\varepsilon}(R) \left[\|\nabla z\|^2 + \|\mathcal{A}^{-1/2} z_t\|^2 \right]. \end{aligned}$$

After selecting appropriate η and ε this implies the desired conclusion in (13) and (14).

We can use (13) and (14) to prove the existence of weak solutions for initial data $(u_0; u_1) \in \mathcal{H}$ by limit transition from smoother solutions. Indeed, we can choose a sequence $(u_0^n; u_1^n)$ elements from $(H^2 \cap H_0^1)(\Omega) \times L_2(\Omega)$ such that $(u_0^n; u_1^n) \rightarrow (u_0; u_1)$ in \mathcal{H} . Due to (13) and (14) the corresponding solutions $u^n(t)$ converge to a function $u(t)$ in the sense that

$$\max_{t \in [0, T]} \{ \|u_t^n(t) - u_t(t)\|_{-1}^2 + \|u^n(t) - u(t)\|_1^2 \} + \int_0^T \|u^n(\tau) - u(\tau)\|_{L_{p+1}(\Omega)}^{p+1} d\tau \rightarrow 0.$$

From the boundedness provided by the energy relation in (10) for u^n we also have $*$ -weak convergence of $(u^n; u_t^n)$ to $(u; u_t)$ in the space

$$L_{\infty}(0, T; H^1(\Omega)) \cap L_{p+1}(\Omega) \times [L_{\infty}(0, T; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega))].$$

This implies that $u(t)$ is a weak solution. By (13) this solution is unique. Moreover, this solution satisfies the corresponding energy *inequality*.

Now we prove smoothness properties of weak solutions stated in (11) using the same method as [23] (see also [2]).

As usual the argument below can be justified by considering Galerkin approximations.

Let $u(t)$ be a solution such that $\|(u(t); u_t(t))\|_{\mathcal{H}} \leq R$ for $t \in [0, T]$. Formal differentiation gives that $v = u_t(t)$ solves the equation

$$v_{tt} - \sigma(\|\nabla u\|^2) \Delta v_t - \phi(\|\nabla u\|^2) \Delta v + f'(u)v + G_*(u, u_t; t) = 0, \quad (40)$$

where

$$G_*(u, u_t; t) = -2 [\sigma'(\|\nabla u\|^2) \Delta u_t + \phi'(\|\nabla u\|^2) \Delta u] (\nabla u, \nabla u_t).$$

Thus, multiplying equation (40) by v we have that

$$\begin{aligned} \frac{d}{dt} \left[(v, v_t) + \frac{1}{2} \sigma(\|\nabla u\|^2) \|\nabla v\|^2 \right] + \phi(\|\nabla u\|^2) \|\nabla v\|^2 + (f'(u)v, v) \\ \leq \|v_t\|^2 + C_R [|(\nabla u, \nabla v)|^2 + |(\nabla u, \nabla v)| \|\nabla v\|^2]. \end{aligned}$$

This implies that

$$\frac{d}{dt} \left[(v, v_t) + \frac{1}{2} \sigma(\|\nabla u\|^2) \|\nabla v\|^2 \right] + c_0 \int_{\Omega} |u|^{p-1} v^2 dx \leq \|v_t\|^2 + C_R [1 + \|\nabla u_t\|] \|\nabla v\|^2,$$

where $c_0 > 0$ in the supercritical case only. Using the multiplier $\mathcal{A}^{-1}v_t$ in (40) we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{A}^{-1/2} v_t\|^2 + \sigma(\|\nabla u\|^2) \|v_t\|^2 \leq C_R \|\nabla v\| \|v_t\| + \left| \int_{\Omega} f'(u) v \mathcal{A}^{-1} v_t dx \right|.$$

As above (cf. (36)) in the supercritical case we have that

$$\left| \int_{\Omega} f'(u) v \mathcal{A}^{-1} v_t dx \right| \leq \varepsilon \int_{\Omega} (1 + |u|^{p-1}) |v|^2 dx + C_{R,\varepsilon} \|\mathcal{A}^{-1} v_t\|_{L_{p+1}}^2.$$

for any $\varepsilon > 0$. Thus

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathcal{A}^{-1/2} v_t\|^2 + \sigma(\|\nabla u\|^2) \|v_t\|^2 \\ \leq \varepsilon \left(\|v_t\|^2 + c_0 \int_{\Omega} |u|^{p-1} v^2 dx \right) + C_{R,\varepsilon} \left[\|\nabla v\|^2 + \|\mathcal{A}^{-1} v_t\|_{L_{p+1}}^2 \right]. \end{aligned}$$

We introduce now the functional

$$\Psi_*(t) = \frac{1}{2} \|\mathcal{A}^{-1/2} v_t\|^2 + \eta \left[(v, v_t) + \frac{1}{2} \sigma(\|\nabla u\|^2) \|\nabla v\|^2 \right]$$

for $\eta > 0$ small enough. It is obvious that for $\eta \leq \eta_0(R)$ we have

$$a_R \eta \left[\|\mathcal{A}^{-1/2} v_t\|^2 + \|\nabla v\|^2 \right] \leq \Psi_*(t) \leq b_R \left[\|\mathcal{A}^{-1/2} v_t\|^2 + \|\nabla v\|^2 \right].$$

Using (33) we also have that

$$\begin{aligned} \frac{d\Psi_*}{dt} + [\sigma(\|\nabla u\|^2) - \eta - \varepsilon] \|v_t\|^2 + c_0 [\eta - \varepsilon] \int_{\Omega} |u|^{p-1} v^2 dx \\ \leq C_{R,\varepsilon} (1 + \|\nabla u_t\|^2) \left[\|\mathcal{A}^{-1/2} v_t\|^2 + \|\nabla v\|^2 \right]. \end{aligned}$$

In particular for $\eta > 0$ small enough, there exists $\alpha_R > 0$ such that

$$\frac{d\Psi_*}{dt} + \alpha_R \left(\|v_t\|^2 + c_0 \int_{\Omega} |u|^{p-1} v^2 dx \right) \leq C_R (1 + \|\nabla u_t\|^2) \Psi_*(t), \quad (41)$$

where $c_0 > 0$ in the supercritical case only. This implies that

$$\begin{aligned} \|u_{tt}(t)\|_{-1}^2 + \|\nabla u_t(t)\|^2 + \int_0^t \left[\|u_{tt}(\tau)\|^2 + c_0 \int_{\Omega} |u(x, \tau)|^{p-1} |u_t(x, \tau)|^2 dx \right] d\tau \\ \leq C_{R,T} (\|u_{tt}(0)\|_{-1}^2 + \|\nabla u_t(0)\|^2) \end{aligned}$$

for $t \in [0, T]$, where $c_0 = 0$ in the non-supercritical case. This formula demonstrates preservation of some smoothness. To obtain (11) we multiply (41) by t^α . This gives us the relation

$$\frac{d}{dt}(t^\alpha \Psi_*) + \alpha_R t^\alpha \|v_t\|^2 \leq C_R (1 + \|\nabla u_t\|^2) [t^\alpha \Psi_*] + \alpha t^{\alpha-1} b_R [\|\mathcal{A}^{-1/2} v_t\|^2 + \|\nabla v\|^2]. \quad (42)$$

One can see that

$$t^{\alpha-1} \|\nabla v\|^2 \leq 1 + t^{2(\alpha-1)} \|\nabla u_t\|^2 \|\nabla v\|^2 \leq C_T [1 + \|\nabla u_t\|^2 (t^\alpha \Psi_*)], \quad t \in [0, T],$$

provided $\alpha \geq 2$. We also have that $\|\mathcal{A}^{-1/2} v_t\|^2 \leq C \|v_t\|^\delta \|\mathcal{A}^{-m} u_{tt}\|^{2-\delta}$ for any $m \geq 1$ with $\delta = \delta(m) \in [1, 2)$. Since

$$\mathcal{A}^{-m} u_{tt} = \sigma(\|\nabla u\|^2) \mathcal{A}^{-m+1} u_t + \phi(\|\nabla u\|^2) \mathcal{A}^{-m+1} u - \mathcal{A}^{-m}(f(u) - h),$$

one can see that $\|\mathcal{A}^{-m} u_{tt}\| \leq C_R + \int_{\Omega} |f(u)| dx \leq \tilde{C}_R$ for $m \geq \max\{1, d/2\}$. Therefore

$$t^{\alpha-1} \|\mathcal{A}^{-1/2} v_t\|^2 \leq C_\delta t^{(\alpha-1)} \|v_t\|^\delta \leq \varepsilon t^\alpha \|v_t\|^2 + C_{R,T,\delta,\varepsilon}, \quad t \in [0, T],$$

provided $2(\alpha-1)/\delta \geq \alpha$. Thus from (42) we have that

$$\frac{d}{dt}(t^\alpha \Psi_*) \leq C_{R,T} + C_{R,T} (1 + \|\nabla u_t\|^2) [t^\alpha \Psi_*].$$

This implies (11) with some $\beta > 0$.

Now we prove that the function $t \mapsto (u(t); u_t(t))$ is (strongly) continuous in $\mathcal{H} = [H_0^1(\Omega) \cap L_{p+1}(\Omega)] \times L_2(\Omega)$ and establish energy relation (12). We concentrate on the supercritical case only (other cases are much simpler).

We first note that the function $t \mapsto (u(t); u_t(t))$ is weakly continuous in \mathcal{H} for every $t \geq 0$ and $t \mapsto u(t)$ is strongly continuous in $H_0^1(\Omega)$, $t \geq 0$. Moreover, (11) implies that $t \mapsto (u(t); u_t(t))$ is continuous in $H_0^1(\Omega) \times L_2(\Omega)$ at every point $t_0 > 0$.

Let us prove that $t \mapsto \|u(t)\|_{L_{p+1}(\Omega)}^{p+1}$ is continuous at $t_0 > 0$. From (11) and from the energy inequality for weak solutions we have that

$$\int_a^b \int_{\Omega} |u|^{p-1} (|u|^2 + |u_t|^2) dx dt \leq C_{a,b}, \quad \text{for all } 0 < a < b \leq T. \quad (43)$$

On smooth functions we also have that

$$\left| \frac{d}{dt} \|u(t)\|_{L_{p+1}(\Omega)}^{p+1} \right| = (p+1) \left| \int_{\Omega} |u|^p u_t dx \right| \leq \frac{p+1}{2} \int_{\Omega} |u|^{p-1} (|u|^2 + |u_t|^2) dx.$$

Therefore by (43) for $t_2 > t_1 > a$ we have that

$$\left| \|u(t_2)\|_{L_{p+1}(\Omega)}^{p+1} - \|u(t_1)\|_{L_{p+1}(\Omega)}^{p+1} \right| \leq \frac{p+1}{2} \int_{t_1}^{t_2} \int_{\Omega} |u|^{p-1} (|u|^2 + |u_t|^2) dx dt \rightarrow 0 \quad \text{as } t_2 - t_1 \rightarrow 0.$$

Thus the function $t \mapsto \|u(t)\|_{L_{p+1}(\Omega)}^{p+1}$ is continuous for $t > 0$. Since $u(t)$ is weakly continuous in $L_{p+1}(\Omega)$ for $t > 0$ and $L_{p+1}(\Omega)$ is uniformly convex, we conclude that $u(t)$ is norm-continuous in $L_{p+1}(\Omega)$ at every point $t_0 > 0$.

In the next step we establish energy relation (12) for every $t > s > 0$. For this we note that by (11) equation (1) is satisfied on any interval $[a, b]$, $0 < a < b \leq T$, as an equality in space $[H^{-1} + L_{1+1/p}](\Omega)$. Moreover one can see that $f(u)u_t \in L_1([a, b] \times \Omega)$. This allows us to multiply equation (1) by u_t and prove (12) for $t \geq s > 0$.

To prove energy relation (12) for $s = 0$ we note that it follows from (12) for $t \geq s > 0$ that the limit $\mathcal{E}(u(s), u_t(s))$ as $s \rightarrow 0$ exists and

$$\mathcal{E}_* \equiv \lim_{s \rightarrow 0} \mathcal{E}(u(s), u_t(s)) = \mathcal{E}(u(t), u_t(t)) + \int_0^t \sigma(\|\nabla u(\tau)\|^2) \|\nabla u_t(\tau)\|^2 d\tau.$$

Since $u(t)$ is continuous in $H_0^1(\Omega)$ on $[0, +\infty)$, we conclude that there is a sequence $\{s_n\}$, $s_n \rightarrow 0$, such that $u(x, s_n) \rightarrow u_0(x)$ almost surely. Since $F(u) \geq -c$ for all $u \in \mathbb{R}$, from Fatou's lemma we have that

$$\int_{\Omega} F(u_0(x)) dx \leq \liminf_{s \rightarrow 0} \int_{\Omega} F(u(x, s)) dx.$$

The property of weak continuity of $u_t(t)$ at zero implies that $\|u_1\|^2 \leq \liminf_{s \rightarrow 0} \|u_t(s)\|^2$. Thus we arrive to the relation $\mathcal{E}(u_0, u_1) \leq \mathcal{E}_*$. Therefore from the energy *inequality* for weak solutions we obtain (12) for all $t \geq s \geq 0$.

No we conclude the proof of strong continuity of $t \mapsto (u(t); u_t(t))$ in \mathcal{H} at $t = 0$. From the continuity of $t \mapsto \mathcal{E}(u(t), \partial_t u(t))$ and property that $u(t) \rightarrow u_0$ in $H_0^1(\Omega)$ as $t \rightarrow 0$ one can see by contradiction that

$$\lim_{t \rightarrow 0} \|u_t(t)\|^2 = \|u_1\|^2, \quad \lim_{t \rightarrow 0} \int_{\Omega} F(u(x, t)) dx = \int_{\Omega} F(u_0(x)) dx.$$

The first relation implies that $u(t)$ is continuous in $L_2(\Omega)$ at $t = 0$. It follows from Assumption 1.1 that

$$|u(x, t)|^{p+1} \leq C_1 F(u(x, t)) + C_2 \quad \text{for almost all } x \in \Omega, t > 0.$$

We also have that $|u(x, t)|^{p+1} \rightarrow |u_0(x)|^{p+1}$ almost everywhere along some sequence as $t \rightarrow 0$. Therefore from the Lebesgue dominated convergence theorem we conclude that

$$\|u(t)\|_{L_{p+1}(\Omega)}^{p+1} \rightarrow \|u_0\|_{L_{p+1}(\Omega)}^{p+1} \quad \text{as } t \rightarrow 0$$

along a subsequence. Using again uniform convexity of the space $L_{p+1}(\Omega)$ we conclude that $u(t)$ is strongly continuous in $L_{p+1}(\Omega)$. The proof of Theorem 2.2 is complete. \square

Remark 2.6 We do not know how to avoid the assumption $p < p_{**} = (d+4)/(d-4)_+$ (which arises in dimension d greater than 4) in the proof of well-posedness. The point is that we cannot use smother multipliers like $\mathcal{A}^{-2l} z_t$ and $\mathcal{A}^{-2l} z$ to achieve the goal because the term $\|\nabla z\|^2$ goes into picture in the estimate for G . If we will use the multipliers $\mathcal{A}^{-2l} z_t$ and z in the proof of uniqueness of solutions, then we get a problem with the corresponding two-sided estimate for the corresponding analog of the function $\Psi(t)$ given by (38).

As for the existence of weak solutions without the requirement $p \geq p_{**}$ in the case $d \geq 4$ we note that the standard a priori estimates for $u^N(t)$ (see (19), (20) and (24)) can be also easily obtained in this case. The main difficulty in this situation is the limit transition in the nonlocal terms $\phi(\|u^N(t)\|^2)$ and $\sigma(\|u^N(t)\|^2)$. To do this we can apply the same procedure as in [5] with $\sigma = \text{const}$, $f(u) \equiv 0$. We do not provide details because we do not know how establish uniqueness for this case.

Remark 2.7 In addition to Assumption 1.1 assume that either

$$\Phi(s) \equiv \int_0^s \phi(\xi) d\xi \rightarrow +\infty \text{ as } s \rightarrow +\infty \text{ and } \mu_f > 0, \quad (44)$$

or else

$$\hat{\mu}_\phi := \liminf_{s \rightarrow +\infty} \phi(s) > 0 \text{ and } \hat{\mu}_\phi \lambda_1 + \mu_f > 0, \quad (45)$$

where μ_f is defined by (4) and λ_1 is the first eigenvalue of the minus Laplace operator in Ω with the Dirichlet boundary conditions (if $\hat{\mu}_\phi = +\infty$, then $\mu_f > -\infty$ can be arbitrary). In this case it is easy to see that (18) holds with $\eta = \nu = 0$. Therefore the energy relation in (12) yields

$$\sup_{t \in \mathbb{R}_+} \mathcal{E}_+^0(u(t), u_t(t)) \leq C_R \text{ provided } \mathcal{E}_+^0(u_0, u_1) \leq R, \quad (46)$$

where $R > 0$ is arbitrary and \mathcal{E}_+^0 is defined by (16) with $\eta = 0$. Now using either (44) of (45) we can conclude from (46) that

$$\sup_{t \in \mathbb{R}_+} \|\nabla u(t)\| \leq C_R \text{ and } \inf_{t \in \mathbb{R}_+} \sigma(\|\nabla u(t)\|^2) \geq \sigma_R > 0. \quad (47)$$

Therefore under the conditions above the energy relation in (12) along with (46) implies that

$$\sup_{t \in \mathbb{R}_+} \mathcal{E}_+^0(u(t), u_t(t)) + \int_0^\infty \|\nabla u_t(\tau)\|^2 d\tau \leq C_R \quad (48)$$

for any initial data such that $\mathcal{E}_+^0(u_0, u_1) \leq R$. We note that in the case considered the energy type function \mathcal{E}_+^0 is topologically equivalent to the norm on \mathcal{H} in the sense that $\mathcal{E}_+^0(u_0, u_1) \leq R$ for some $R > 0$ if and only if $\|(u_0; u_1)\|_{\mathcal{H}} \leq R_*$ for some $R_* > 0$.

3 Long-time dynamics

3.1 Generation of an evolution semigroup

By Theorem 2.2 problem (1) generates an evolution semigroup $S(t)$ in the space \mathcal{H} by the formula

$$S(t)y = (u(t); \partial_t u(t)), \text{ where } y = (u_0; u_1) \in \mathcal{H} \text{ and } u(t) \text{ solves (1)} \quad (49)$$

To describe continuity properties of $S(t)$ it is convenient to introduce the following notion.

Definition 3.1 (Partially strong topology) A sequence $\{(u_0^n; u_1^n)\} \subset \mathcal{H}$ is said to be *partially strongly convergent* to $(u_0; u_1) \in \mathcal{H}$ if $u_0^n \rightarrow u_0$ strongly in $H_0^1(\Omega)$, $u_0^n \rightarrow u_0$ weakly in $L_{p+1}(\Omega)$ and $u_1^n \rightarrow u_1$ strongly in $L_2(\Omega)$ as $n \rightarrow \infty$ (in the case when $d \leq 2$ we take $1 < p < \infty$ arbitrary).

It is obvious that the partially strong convergence becomes strong in the non-supercritical case ($H_0^1(\Omega) \subset L_{p+1}(\Omega)$).

Proposition 3.2 *Let Assumption 1.1 be in force. Then the evolution semigroup $S(t)$ given by (49) is a continuous mapping in \mathcal{H} with respect to the strong topology. Moreover,*

(A) General case: *For every $t > 0$ $S(t)$ maps \mathcal{H} into itself continuously in the partially strong topology.*

(B) Non-supercritical case ((6) fails): *For any $R > 0$ and $T > 0$ there exists $a_{R,T} > 0$ such that*

$$\|S(t)y_1 - S(t)y_2\|_{\mathcal{H}} \leq a_{R,T}\|y_1 - y_2\|_{\mathcal{H}}, \quad t \in [0, T],$$

for all $y_1, y_2 \in \mathcal{H} = H_0^1(\Omega) \times L_2(\Omega)$ such that $\|y_i\| \leq R$. Thus, in this case $S(t)$ is a locally Lipschitz continuous mapping in \mathcal{H} with respect to the strong topology.

Proof. Let $(u_0^n; u_1^n) \rightarrow (u_0; u_1)$ in \mathcal{H} as $n \rightarrow \infty$. From the energy relation we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\mathcal{E}(u^n(t), u_t^n(t)) + \int_0^t \sigma(\|\nabla u^n(\tau)\|^2) \|\nabla u_t^n(\tau)\|^2 d\tau \right] &= \lim_{n \rightarrow \infty} \mathcal{E}(u_0^n, u_1^n) \\ &= \mathcal{E}(u_0, u_1) = \mathcal{E}(u(t), u_t(t)) + \int_0^t \sigma(\|\nabla u(\tau)\|^2) \|\nabla u_t(\tau)\|^2 d\tau, \end{aligned} \quad (50)$$

where $u^n(t)$ and $u(t)$ are weak solutions with initial data $(u_0^n; u_1^n)$ and $(u_0; u_1)$. Using (13) and the low continuity property of weak convergence one can see from (50) that $u^n(t) \rightarrow u(t)$ in $H_0^1(\Omega)$ and also

$$\lim_{n \rightarrow \infty} \left[\frac{1}{2} \|u_t^n(t)\|^2 + \int_{\Omega} F(u^n(x, t)) dx \right] = \frac{1}{2} \|u_t(t)\|^2 + \int_{\Omega} F(u(x, t)) dx.$$

As in the proof of the strong time continuity of weak solutions in Theorem 2.2 this allows us to obtain the strong continuity with respect to initial data.

Now we establish additional continuity properties stated in **(A)** and **(B)**.

(A) This easily follows from uniform boundedness of $\|u_t^n(t)\|$ and $\|u^n(t)\|_{L_{p+1}(\Omega)}$ on each interval $[0, T]$ (which implies the corresponding weak compactness) and from Lipschitz type estimate in (13) for the difference of two solutions. We also use the fact that by (11) $\|\nabla u_t^n(t)\|$ is uniformly bounded for each $t > 0$.

(B) Let $S(t)y_i = (u^i(t); u_t^i(t))$, $i = 1, 2$. Then in the non-supercritical case we have (31). Therefore using (10) and Lemma 2.3 we obtain that

$$\frac{d}{dt} \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \|\nabla z\|^2 \right] \leq \|z_t\|^2 + C_{R,T} (1 + \|\nabla u_t^1\| + \|\nabla u_t^2\|) \|\nabla z\|^2,$$

where $z = u^1 - u^2$ and $\sigma_{12}(t)$ is defined in Lemma 2.3.

In the case considered we can multiply equation (28) by z_t and obtain that

$$\frac{1}{2} \frac{d}{dt} \|z_t\|^2 + \frac{1}{2} \sigma_{12}(t) \|\nabla z_t\|^2 + G(t) = -\frac{1}{2} \phi_{12}(t) (\nabla z, \nabla z_t) \leq C_{R,T} \|\nabla z_t\| \|\nabla z\| \quad (51)$$

Here above

$$G(t) \equiv (G(u^1, u^2; t), z_t) = H_1(t) + H_2(t) + H_3(t), \quad (52)$$

where

$$H_1(t) = \frac{1}{2}[\sigma_1(t) - \sigma_2(t)](\nabla(u_t^1 + u_t^2), \nabla z_t),$$

$$H_2(t) = \tilde{\phi}_{12}(t)(\nabla(u^1 + u^2), \nabla z)(\nabla(u^1 + u^2), \nabla z_t)$$

with $\tilde{\phi}_{12}(t)$ given by (27), and $H_3(t) = (f(u^1) - f(u^2), z_t)$. Using these representations one can see that

$$\begin{aligned} |(G(u^1, u^2; t), z_t)| &\leq C_{R,T}(1 + \|\nabla u_t^1\| + \|\nabla u_t^2\|)\|\nabla z_t\|\|\nabla z\| \\ &\leq \varepsilon\|\nabla z_t\|^2 + C_{R,T,\varepsilon}(1 + \|\nabla u_t^1\|^2 + \|\nabla u_t^2\|^2)\|\nabla z\|^2. \end{aligned}$$

for any $\varepsilon > 0$. Therefore the function

$$V(t) = \frac{1}{2}\|z_t\|^2 + \eta \left[(z, z_t) + \frac{1}{4}\sigma_{12}(t)\|\nabla z\|^2 \right]$$

for $\eta > 0$ small enough satisfies the relations

$$a_{R,T} [\|z_t\|^2 + \|\nabla z\|^2] \leq V(t) \leq b_{R,T} [\|z_t\|^2 + \|\nabla z\|^2]$$

and

$$\frac{d}{dt}V(t) \leq c_{R,T}(1 + \|\nabla u_t^1\|^2 + \|\nabla u_t^2\|^2)V(t)$$

with positive constants $a_{R,T}$, $b_{R,T}$ and $c_{R,T}$. Thus Gronwall's lemma and the finiteness of the dissipation integral in (10) imply the desired conclusion. \square

Remark 3.3 One can see from the energy relation in (12) that the dynamical system generated by semigroup $S(t)$ is gradient on \mathcal{H} (with respect to the strong topology), i.e., there exists a continuous functional $\Psi(y)$ on \mathcal{H} (called a *strict Lyapunov function*) possessing the properties (i) $\Psi(S(t)y) \leq \Psi(y)$ for all $t \geq 0$ and $y \in \mathcal{H}$; (ii) equality $\Psi(y) = \Psi(S(t)y)$ may take place for all $t > 0$ if only y is a stationary point of $S(t)$. In our case the full energy $\mathcal{E}(u_0; u_1)$ is a strict Lyapunov function.

3.2 Dissipativity

Now we establish some dissipativity properties of the semigroup $S(t)$. Fro this we need the following hypothesis.

Assumption 3.4 We assume³ that either (45) holds or else

$$\phi(s)s \rightarrow +\infty \text{ as } s \rightarrow +\infty \text{ and } \mu_f = \liminf_{|s| \rightarrow \infty} \{s^{-1}f(s)\} > 0. \quad (53)$$

Proposition 3.5 *Let Assumptions 1.1 and 3.4 be in force. Then there exists $R_* > 0$ such that for any $R > 0$ we can find $t_R \geq 0$ such that*

$$\|(u(t); u_t(t))\|_{\mathcal{H}} \leq R_* \text{ for all } t \geq t_R,$$

where $u(t)$ is a solution to (1) with initial data $(u_0; u_1) \in \mathcal{H}$ such that $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$. In particular, the evolution semigroup $S(t)$ is dissipative in \mathcal{H} and

$$\mathcal{B}_* = \{(u_0; u_1) \in \mathcal{H} : \|(u_0; u_1)\|_{\mathcal{H}} \leq R_*\} \text{ is an absorbing set.} \quad (54)$$

³Under these additional conditions the properties in (2) and (3) holds automatically with $\eta_0 = c_1 = 0$.

Proof. Let $u(t)$ be a solution to (1) with initial data possessing the property $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$. Multiplying equation (1) by u we obtain that

$$\frac{d}{dt} \left[(u, u_t) + \frac{1}{2} \Sigma(\|\nabla u\|^2) \right] - \|u_t\|^2 + \phi(\|\nabla u\|^2) \|\nabla u\|^2 + (f(u), u) - (h, u) = 0,$$

where $\Sigma(s) = \int_0^s \sigma(\xi) d\xi$. Therefore using the energy relation in (12) for the function $W(t) = \mathcal{W}^{\eta, 0}(u(t), u_t(t))$ with $\mathcal{W}^{\eta, \nu}$ given by (17) we obtain that

$$\frac{d}{dt} W(t) + \sigma(\|\nabla u\|^2) \|\nabla u_t\|^2 - \eta \|u_t\|^2 + \eta \phi(\|\nabla u\|^2) \|\nabla u\|^2 + \eta (f(u), u) - \eta (h, u) = 0.$$

Since (53) implies (44), we have (47). By (4) and (6) we have that

$$(u, f(u)) \geq d_0 \|u\|_{L_{p+1}(\Omega)}^{p+1} + d_1 (\mu_f - \delta) \|u\|^2 - d_2(\delta), \quad \forall \delta > 0,$$

where $d_0 > 0$, $d_1 = 0$ in the supercritical case and $d_0 = 0$, $d_1 = 1$ in other cases. In both cases (either (45) or (53)) this yields

$$\frac{d}{dt} W(t) + (\sigma_R - \eta) \|u_t\|^2 + \eta c_0 \phi(\|\nabla u\|^2) \|\nabla u\|^2 + \frac{\eta d_0}{2} \|u\|_{L_{p+1}(\Omega)}^{p+1} + \eta c_1 \|u\|^2 \leq \eta c_2$$

with positive c_i independent of R and $d_0 > 0$ in the supercritical case only. Thus there exist constants $a_0, a_1 > 0$ independent of R and also $0 < \eta_R \leq 1$ such that

$$\frac{d}{dt} \mathcal{W}^{\eta, 0}(u(t), u_t(t)) + \eta a_0 \left[\|u_t\|^2 + \phi(\|\nabla u\|^2) \|\nabla u\|^2 + d_0 \|u\|_{L_{p+1}(\Omega)}^{p+1} + \|u\|^2 \right] \leq \eta a_1,$$

for all initial data $(u_0; u_1) \in \mathcal{H}$ such that $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$ and for each $0 < \eta \leq \eta_R$. Moreover, for this choice of η we have relation (18) with $\nu = 0$ and $a(\eta) \geq a(0)$. Therefore using the "barrier" method (see, e.g., [9, Theorem 1.4.1] and [24, Theorem 2.1]) we can conclude the proof. \square

Remark 3.6 Let $\mathcal{B}_0 = \left[\bigcup_{t \geq 1+t_*} S(t) \mathcal{B}_* \right]_{ps}$, where \mathcal{B}_* is given by (54), $t_* \geq 0$ is chosen such that $S(t) \mathcal{B}_* \subset \mathcal{B}_*$ for $t \geq t_*$ and $[\cdot]_{ps}$ denotes the closure in the partially strong topology. By the standard argument (see, e.g., [44]) one can see that \mathcal{B}_0 is a closed forward invariant bounded absorbing set which lies in \mathcal{B}_* . Moreover, by (11) the set \mathcal{B}_0 is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$.

For a strictly positive stiffness coefficient we can also prove a dissipativity property in the space $\mathcal{H}_* = (H^2 \cap H_0^1)(\Omega) \times L_2(\Omega)^4$. Indeed, we have the following assertion.

Proposition 3.7 *In addition to the hypotheses of Proposition 3.5 we assume that $\phi(s)$ is strictly positive (i.e., $\phi(s) \geq \phi_0 > 0$ for all $s \in \mathbb{R}_+$) and $f'(s) \geq -c$ for all $s \in \mathbb{R}$ in the case when (5) holds with $p = p_*$. Let $u(t)$ be a solution to (1) with initial data $(u_0; u_1) \in \mathcal{H}$ such that $u_0 \in H^2(\Omega)$ and $\|(u_0; u_1)\|_{\mathcal{H}} \leq R$ for some R . Then there exist $B > 0$ and $\gamma > 0$ independent of R and $C_R > 0$ such that*

$$\|\Delta u(t)\|^2 \leq C_R (1 + \|\Delta u_0\|^2) e^{-\gamma t} + B \quad \text{for all } t \geq 0. \quad (55)$$

⁴We note that $\mathcal{H}_* \subset \mathcal{H}$ because $H^2(\Omega) \subset L_{p+1}(\Omega)$ for $p < p_{**}$.

Proof. By Proposition 3.5 we have that $\|(u(t); u_t(t))\| \leq R_*$ for all $t \geq t_R$. Therefore it follows from (21) that

$$\frac{d}{dt}\chi(t) + \frac{\phi_0}{2}\|\Delta u(t)\|^2 \leq \|\nabla u_t(t)\|^2 + C_{R_*}\|\nabla u_t(t)\|^2\|\Delta u(t)\|^2 + C_{R_*} \quad \text{for all } t \geq t_R,$$

where $\chi(t) = -(u_t(t), \Delta u(t)) + \sigma(\|\nabla u(t)\|^2)\|\Delta u(t)\|^2/2$. One can see that

$$a_1\|\Delta u(t)\|^2 - a_2 \leq \chi(t) \leq a_3\|\Delta u(t)\|^2 + a_4 \quad \text{for all } t \geq t_R, \quad (56)$$

where $a_i = a_i(R_*)$ are positive constants. Therefore using the finiteness of the dissipation integral $\int_{t_R}^\infty \|\nabla u_t(t)\|^2 dt < C_{R_*}$ we can conclude that

$$\chi(t) \leq C_R|\chi(t_R)|e^{-\gamma(t-t_R)} + C_{R_*} \quad \text{for all } t \geq t_R.$$

Thus (56) and (15) yield (55). \square

Remark 3.8 Using (15) one can show that the evolution operator $S(t)$ generated by (1) maps the space $\mathcal{H}_* = (H^2 \cap H_0^1)(\Omega) \times L^2(\Omega)$ into itself and weakly continuous with respect to t and initial data. Therefore under the hypotheses of Proposition 3.7 by [2, Theorem 1, Sect.II.2] $S(t)$ possesses a weak global attractor in \mathcal{H}_* . Unfortunately we cannot derive from Proposition 3.5 a similar result in the space \mathcal{H} because we cannot prove that $S(t)$ is a weakly closed mapping in \mathcal{H} (a mapping $S : \mathcal{H} \mapsto \mathcal{H}$ is said to be weakly closed if weak convergences $u_n \rightarrow u$ and $Su_n \rightarrow v$ imply $Su = v$). Below we prove the existence of a global attractor in \mathcal{H} under additional hypotheses concerning the stiffness coefficient ϕ .

3.3 Asymptotic compactness

In this section we prove several properties of asymptotic compactness of the semigroup $S(t)$.

We start with the following theorem.

Theorem 3.9 *Let Assumptions 1.1 and 3.4 be in force. Assume also that $\phi(s)$ is strictly positive (i.e., $\phi(s) > 0$ for all $s \in \mathbb{R}_+$) and $f'(s) \geq -c$ for all $s \in \mathbb{R}$ in the non-supercritical case (the bounds in (6) are not valid). Then there exists a bounded set \mathcal{K} in the space $\mathcal{H}_1 = (H^2 \cap H_0^1)(\Omega) \times H_0^1(\Omega)$ and the constants $C, \gamma > 0$ such that*

$$\sup \left\{ \text{dist}_{H_0^1(\Omega) \times H_0^1(\Omega)}(S(t)y, \mathcal{K}) : y \in B \right\} \leq Ce^{-\gamma(t-t_B)}, \quad t \geq t_B, \quad (57)$$

for any bounded set B from \mathcal{H} . Moreover, we have that $\mathcal{K} \subset \mathcal{B}_0$, where \mathcal{B}_0 is the positively invariant set constructed in Remark 3.6.

Proof. We use a splitting method relying on the idea presented in [38] (see also [23]).

We first note that it is sufficient to prove (57) for $B = \mathcal{B}_0$, where $\mathcal{B}_0 \subset \mathcal{H} \cap (H_0^1 \times H_0^1)(\Omega)$ is the invariant absorbing set constructed in Remark 3.6.

From (11) and (48) we obviously have that

$$\|\nabla u(t)\|^2 + \|\nabla u_t(t)\|^2 + \int_t^{t+1} \|u_{tt}(\tau)\|^2 d\tau + \int_0^\infty \|\nabla u_t(\tau)\|^2 d\tau \leq C_{\mathcal{B}_0}, \quad t \geq 0, \quad (58)$$

for any solution $u(t)$ with initial data $(u_0; u_1)$ from \mathcal{B}_0 . Thus we need only to show that there exists a ball $B = \{u \in (H^2 \cap H_0^1)(\Omega) : \|\Delta u\| \leq \rho\}$ which attracts in $H_0^1(\Omega)$ any solution $u(t)$ satisfying (58) with uniform exponential rate.

We denote $\sigma(t) = \sigma(\|\nabla u(t)\|^2)$ and $\phi(t) = \phi(\|\nabla u(t)\|^2)$. Since both σ and ϕ are strictly positive, we have that

$$0 < c_1 \leq \sigma(s), \phi(s) \leq c_2, \quad t \geq 0,$$

where the constants c_1 and c_2 depend only on the size of the absorbing set \mathcal{B}_0 . Let $\nu > 0$ be a parameter (which we choose large enough). Assume that $w(t)$ solves the problems

$$\begin{cases} -\sigma(t)\Delta w_t - \phi(t)\Delta w + \nu w + f(w) = h_u(t) \equiv -u_{tt} + \nu u + h(x), & x \in \Omega, t > 0, \\ w|_{\partial\Omega} = 0, & w(0) = 0. \end{cases} \quad (59)$$

Then one can see that $v(t) = w(t) - u(t)$ satisfies the equation

$$\begin{cases} -\sigma(t)\Delta v_t - \phi(t)\Delta v + \nu v + f(w+v) - f(w) = 0, & x \in \Omega, t > 0, \\ v|_{\partial\Omega} = 0, & v(0) = u_0. \end{cases} \quad (60)$$

As in the proof of Proposition 3.7 using the multiplier $-\Delta w$ in (59) one can see that

$$\frac{1}{2} \frac{d}{dt} [\sigma(t)\|\Delta w(t)\|^2] + \phi(t)\|\Delta w(t)\|^2 \leq [\varepsilon + C_\varepsilon \|\nabla u_t(t)\|^2] \|\Delta w(t)\|^2 + C_\varepsilon \|h_u(t)\|^2$$

for all $t > 0$. Therefore using Gronwall's type argument and the bounds in (58) we obtain that

$$\|\Delta w(t)\|^2 \leq C \int_0^t e^{-\gamma(t-\tau)} \|h_u(\tau)\|^2 d\tau \leq C_{\mathcal{B}_0}, \quad \forall t \geq 0. \quad (61)$$

where $C_{\mathcal{B}_0} > 0$ does not depends on t .

Multiplying (60) by v in a similar way we obtain

$$\frac{1}{2} \frac{d}{dt} [\sigma(t)\|\nabla v(t)\|^2] + \phi(t)\|\nabla v(t)\|^2 \leq [\varepsilon + C_\varepsilon \|\nabla u_t(t)\|^2] \|\nabla v(t)\|^2, \quad t \geq 0,$$

which implies that

$$\|\nabla v(t)\|^2 \leq C \|\nabla u(0)\|^2 e^{-2\gamma t}, \quad t \geq 0. \quad (62)$$

Let $\mathcal{B} = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : \|\Delta u\|^2 \leq C_{\mathcal{B}_0}\}$, where $C_{\mathcal{B}_0}$ is the constant from (61). It follows from (61) and (62) that

$$\text{dist}_{H_0^1(\Omega)}(u(t), \mathcal{B}) = \inf_{b \in \mathcal{B}} \|w(t) + v(t) - b\|_1 \leq \|v(t)\|_1 \leq C e^{-\gamma t}, \quad t \geq 0. \quad (63)$$

This implies the existence of the set \mathcal{K} desired in the statement of the Theorem 3.9. \square

Now we consider the set \mathcal{B}_0 defined in Remark 3.6 as a topological space equipped with the partially strong topology (see Definition 3.1). Since \mathcal{B}_0 bounded in $\mathcal{H} \cap (H_0^1 \times H_0^1)(\Omega)$, this topology can be defined by the metric

$$\mathcal{R}(y, y^*) = \|u_0 - u_0^*\|_1 + \|u_1 - u_1^*\| + \sum_{n=1}^{\infty} 2^{-n} \frac{|(u_0 - u_0^*, g_n)|}{1 + |(u_0 - u_0^*, g_n)|} \quad (64)$$

for $y = (u_0; u_1)$ and $y^* = (u_0^*; u_1^*)$ from \mathcal{B}_0 , where $\{g_n\}$ is a sequence in $L_{(p+1)/p}(\Omega) \cap H^{-1}(\Omega)$ such that $\|g_n\|_{-1} = 1$ and $\text{Span}\{g_n : n \in \mathbb{N}\}$ is dense in $L_{(p+1)/p}(\Omega)$.

Corollary 3.10 *Let the hypotheses of Theorem 3.9 be in force and \mathcal{K} and \mathcal{B}_0 be the same sets as in Theorem 3.9. Then there exist $C, \gamma > 0$ such that*

$$\sup \left\{ \inf_{z \in \mathcal{K}} \mathcal{R}(S(t)y, z) : y \in \mathcal{B}_0 \right\} \leq C e^{-\gamma t} \quad \text{for all } t \geq 0. \quad (65)$$

Proof. As in the proof of Theorem 3.9 using the splitting given by (59) and (60) we have that

$$\begin{aligned} \inf_{z \in \mathcal{K}} \mathcal{R}(S(t)y, z) &\leq \|v(t)\|_1 + \sum_{n=1}^{\infty} 2^{-n} \frac{|(v(t), g_n)|}{1 + |(v(t), g_n)|} \leq \|v(t)\|_1 + 2^{-N+1} + \sum_{n=1}^N 2^{-n} \frac{|(v(t), g_n)|}{1 + |(v(t), g_n)|} \\ &\leq \|v(t)\|_1 \left[1 + \sum_{n=1}^N 2^{-n} \|g_n\|_{-1} \right] + 2^{-N+1} \leq 2\|v(t)\|_1 + 2^{-N+1} \end{aligned}$$

for every $N \in \mathbb{N}$, where $S(t)y = (u(t); u_t(t))$ with $y = (u_0; u_1) \in \mathcal{B}_0$, and v solves (60). We can choose $N = [t]$, where $[t]$ denotes integer part of t . Thus (65) follows from (63). \square

3.4 Global attractor in partially strong topology

We recall the notion of a global attractor and some dynamical characteristics for the semigroup $S(t)$ which depend on a choice of the topology in the phase space (see, e.g., [2, 9, 21, 44] for the general theory).

A bounded set $\mathfrak{A} \subset \mathcal{H}$ is said to be a *global partially strong attractor* for $S(t)$ if (i) \mathfrak{A} is closed with respect to the partially strong (see Definition 3.1) topology, (ii) \mathfrak{A} is strictly invariant ($S(t)\mathfrak{A} = \mathfrak{A}$ for all $t > 0$), and (iii) \mathfrak{A} uniformly attracts in the partially strong topology all other bounded sets: for any (partially strong) vicinity \mathcal{O} of \mathfrak{A} and for any bounded set B in \mathcal{H} there exists $t_* = t_*(\mathcal{O}, B)$ such that $S(t)B \subset \mathcal{O}$ for all $t \geq t_*$.

Fractal dimension $\dim_f^X M$ of a compact set M in a complete metric space X is defined as

$$\dim_f^X M = \limsup_{\varepsilon \rightarrow 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $N(M, \varepsilon)$ is the minimal number of closed sets in X of diameter 2ε which cover M .

We also recall (see, e.g., [2]) that the *unstable set* $\mathbb{M}_+(\mathcal{N})$ emanating from some set $\mathcal{N} \subset \mathcal{H}$ is a subset of \mathcal{H} such that for each $z \in \mathbb{M}_+(\mathcal{N})$ there exists a full trajectory $\{y(t) : t \in \mathbb{R}\}$ satisfying $u(0) = z$ and $\text{dist}_{\mathcal{H}}(y(t), \mathcal{N}) \rightarrow 0$ as $t \rightarrow -\infty$.

Our first main result in this section is the following theorem.

Theorem 3.11 *Let Assumptions 1.1 and 3.4 be in force. Assume also that (i) $\phi(s)$ is strictly positive (i.e., $\phi(s) > 0$ for all $s \in \mathbb{R}_+$) and (ii) $f'(s) \geq -c$ for all $s \in \mathbb{R}$ in the non-super critical case (when (6) does not hold). Then the semigroup $S(t)$ given by (49) possesses a global partially strong attractor \mathfrak{A} in the space \mathcal{H} . Moreover, $\mathfrak{A} \subset \mathcal{H}_1 = [H^2 \cap H_0^1](\Omega) \times H_0^1(\Omega)$ and*

$$\sup_{t \in \mathbb{R}} \left(\|\Delta u(t)\|^2 + \|\nabla u_t(t)\|^2 + \|u_{tt}(t)\|_{-1}^2 + \int_t^{t+1} \|u_{tt}(\tau)\|^2 d\tau \right) \leq C_{\mathfrak{A}} \quad (66)$$

for any full trajectory $\gamma = \{(u(t); u_t(t)) : t \in \mathbb{R}\}$ from the attractor \mathfrak{A} . We also have that

$$\mathfrak{A} = \mathbb{M}_+(\mathcal{N}), \quad \text{where } \mathcal{N} = \{(u; 0) \in \mathcal{H} : \phi(\|\mathcal{A}^{1/2}u\|^2)\mathcal{A}u + f(u) = h\}. \quad (67)$$

Proof. Since \mathcal{B}_0 is an absorbing positively invariant set (see Remark 3.6), to prove the theorem it is sufficient to consider the restriction of $S(t)$ on the metric space \mathcal{B}_0 endowed with the metric \mathcal{R} given by (64). By Corollary 3.10 the dynamical system $(\mathcal{B}_0, S(t))$ is asymptotically compact. Thus (see, e.g., [2, 8, 44]) this system possesses a compact (with respect to the metric \mathcal{R}) global attractor \mathfrak{A} which belongs to \mathcal{H} . It is clear that \mathfrak{A} is a global partially strong attractor for $(\mathcal{H}, S(t))$ with the regularity properties stated in (66).

The attractor \mathfrak{A} is a strictly invariant compact set in \mathcal{H} . By Remark 3.3 the semigroup $S(t)$ is gradient on \mathfrak{A} . Therefore the standard results on gradient systems with compact attractors (see, e.g., [2, 9, 44]) yields (67). Thus the proof of Theorem 3.11 is complete. \square

To obtain the result on dimension for the attractor \mathfrak{A} we need the following amplification of the requirements listed in the first part of Assumption 1.1.

Assumption 3.12 The functions σ and ϕ belong to $C^1(\mathbb{R}_+)$ and possess the properties:

- (i) $\sigma(s) > 0$ and $\phi(s) > 0$ for all $s \in \mathbb{R}_+$;
- (ii) $\lambda_1 \hat{\mu}_\phi + \mu_f > 0$, where $\hat{\mu}_\phi$ is defined in (45), μ_f is given by (4) and λ_1 is the first eigenvalue of the minus Laplace operator in Ω with the Dirichlet boundary conditions (in the supercritical case this requirement holds automatically).

Theorem 3.13 *Let Assumptions 1.1(ii), 3.4 and 3.12 be in force and $\inf_{s \in \mathbb{R}} f'(s) > -\infty$ in the non-supercritical case. Then the global partially strong attractor \mathfrak{A} given by Theorem 3.11 has a finite fractal dimension as a compact set in $\mathcal{H}_r := [H^{1+r} \cap H_0^1](\Omega) \times H^r(\Omega)$ for every $r < 1$.*

Our main ingredient of the proof is the following weak quasi-stability estimate.

Proposition 3.14 (Weak quasi-stability) *Assume that the hypotheses of Theorem 3.13 are in force. Let $u^1(t)$ and $u^2(t)$ be two weak solutions such that $\|u^i(t)\|_2^2 + \|u_t^i(t)\|_1^2 \leq R^2$, for all $t \geq 0$, $i = 1, 2$. Then their difference $z(t) = u^1(t) - u^2(t)$ satisfies the relation*

$$\begin{aligned} \|z_t(t)\|_{-1}^2 + \|\nabla z(t)\|^2 &\leq a_R (\|z_t(0)\|_{-1}^2 + \|\nabla z(0)\|^2) e^{-\gamma_R t} \\ &\quad + b_R \int_0^t e^{-\gamma_R(t-\tau)} \left[\|z(\tau)\|^2 + \|\mathcal{A}^{-l} z_t(\tau)\|^2 \right] d\tau, \end{aligned} \quad (68)$$

where a_R, b_R, γ_R are positive constants and $l \geq 1/2$ can be taken arbitrary.

Proof. Our additional hypothesis on ϕ and also the bounds for solutions u^i imposed allow us to improve the argument which led to (13).

Since

$$|\tilde{\phi}_{12}(t)| |(\nabla(u^1 + u^2), \nabla z)|^2 \leq C_R \|z\|^2, \quad t \geq 0, \quad (69)$$

for our case, it follows from Lemmas 2.3 and 2.5 that

$$\begin{aligned} \frac{d}{dt} \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \cdot \|\nabla z\|^2 \right] &+ \frac{1}{2} \phi_{12}(t) \cdot \|\nabla z\|^2 + c_0 \left[\int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \right] \\ &\leq \|z_t\|^2 + C_R (\|\nabla u_t^1\| + \|\nabla u_t^2\|) \|\nabla z\|^2 + C \|z\|^2, \end{aligned} \quad (70)$$

where $c_0 = 0$ in the non-supercritical case. Now as in the proof of Theorem 2.2 we use the multiplier $\mathcal{A}^{-1}z_t$. However now our considerations of the term $|(G(u^1, u^2; t), \mathcal{A}^{-1}z_t)|$ of the form (35) involves the additional positivity type requirement imposed on ϕ .

Using the inequality $\|\mathcal{A}^{-1/2}z_t\|^2 \leq \eta\|z_t\|^2 + C_\eta\|\mathcal{A}^{-l}z_t\|^2$ for any $\eta > 0$ and $l \geq 1/2$, one can see that

$$|G_1(t)| \leq \varepsilon\|z_t\|^2 + C_{R,\varepsilon} (\|\nabla u_t^1\|^2 + \|\nabla u_t^2\|^2) \|\nabla z\|^2$$

and also, involving (69),

$$|G_2(t)| \leq \varepsilon\|z_t\|^2 + C_{R,\varepsilon} [\|\mathcal{A}^{-l}z_t\|^2 + \|z\|^2]$$

for any $\varepsilon > 0$ and for every $l \geq 1/2$. Therefore from (36) we obtain that

$$\begin{aligned} |(G(u^1, u^2; t), \mathcal{A}^{-1}z_t)| &\leq C_{R,\varepsilon} [(\|\nabla u_t^1\|^2 + \|\nabla u_t^2\|^2) \|\nabla z\|^2 + \|z\|^2 + \|\mathcal{A}^{-l}z_t\|^2] \\ &\quad + \varepsilon \left[\|z_t\|^2 + c_0 \int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \right] \end{aligned}$$

for any $\varepsilon > 0$, where $c_0 = 0$ in the non-supercritical case. Consequently by (34) and (70) the function $\Psi(t)$ given by (38) satisfies the relation

$$\begin{aligned} \frac{d\Psi}{dt} + \frac{\eta}{2}\phi_{12}(t) \cdot \|\nabla z\|^2 + \left[\frac{1}{2}\sigma_{12}(t) - \eta - \varepsilon \right] \|z_t\|^2 \\ + c_0(\eta - \varepsilon) \int_{\Omega} (|u^1|^{p-1} + |u^2|^{p-1}) |z|^2 dx \leq C_\varepsilon(R) [d_{12}(t)\|\nabla z\|^2 + \|\mathcal{A}^{-l}z_t\|^2 + \|z\|^2], \end{aligned}$$

where $d_{12}(t) = \|\nabla u_t^1(t)\|^2 + \|\nabla u_t^2(t)\|^2$. Therefore after an appropriate choice of η and ε we have that

$$\frac{d\Psi}{dt} + \alpha_{12}(t)\Psi \leq c_R [\|\mathcal{A}^{-l}z_t\|^2 + \|z\|^2] \quad \text{with} \quad \alpha_{12}(t) = \frac{\eta}{2}\phi_{12}(t) - c_R d_{12}(t),$$

This implies that

$$\Psi(t) \leq c_R \exp \left\{ -\int_0^t \alpha_{12}(\tau) d\tau \right\} \Psi(0) + c_R \int_0^t \exp \left\{ -\int_\tau^t \alpha_{12}(\xi) d\xi \right\} [\|\mathcal{A}^{-l}z_t(\tau)\|^2 + \|z(\tau)\|^2] d\tau. \quad (71)$$

Under Assumption 3.12 by Remark 2.7 we have estimate (48) which yields that

$$\int_\tau^t \alpha_{12}(\xi) d\xi \geq \eta\phi_R \cdot (t - \tau) - c_R \int_\tau^t d_{12}(\xi) d\xi \geq \eta\phi_R \cdot (t - \tau) - C_R$$

for all $t > \tau \geq 0$. with positive ϕ_R and C_R . Thus from (71) and (39) we obtain (68). \square

Lemma 3.15 *Let the hypotheses of Proposition 3.14 be in force. Then the difference $z(t) = u^1(t) - u^2(t)$ of two weak solutions satisfies the relation*

$$\int_0^T \|\mathcal{A}^{-l}z_{tt}(\tau)\|^2 d\tau \leq C_R (\|z_t(0)\|_{-1}^2 + \|\nabla z(0)\|^2) + C_R T \int_0^T [\|z(\tau)\|^2 + \|\mathcal{A}^{-l}z_t(\tau)\|^2] d\tau \quad (72)$$

for every $T \geq 1$, where $C_R > 0$ is a constant and $l \geq 3/2$ is arbitrary such that $L_1(\Omega) \subset H^{-2l}(\Omega)$, i.e. $l > d/4$.

Proof. It follows from (28) that $\|\mathcal{A}^{-l}z_{tt}\| \leq C_R(\|\mathcal{A}^{-l+1}z\| + \|\mathcal{A}^{-l+1}z_t\|) + \|\mathcal{A}^{-l}G(u^1, u^2; t)\|$. By the embedding $L_1(\Omega) \subset H^{-2l}(\Omega)$ we obviously have that

$$\begin{aligned} \|\mathcal{A}^{-l}G(u^1, u^2; t)\| &\leq C_R\|\mathcal{A}^{1/2}z\| + C \int_{\Omega} |f(u^1) - f(u^2)| dx \\ &\leq C_R\|\mathcal{A}^{1/2}z\| + C \int_{\Omega} (1 + |u^1|^{p-1} + |u^2|^{p-1}) |z| dx. \end{aligned}$$

Therefore using (13) (and also (14) in the supercritical case) we obtain that

$$\int_a^b \|\mathcal{A}^{-l}z_{tt}(\tau)\|^2 d\tau \leq C_R (\|z_t(a)\|_{-1}^2 + \|\nabla z(a)\|^2)$$

for every $a < b$ such that $b - a \leq 1$. Therefore

$$\begin{aligned} \int_0^T \|\mathcal{A}^{-l}z_{tt}(\tau)\|^2 d\tau &\leq \sum_{k=0}^{[T]-1} \int_k^{k+1} \|\mathcal{A}^{-l}z_{tt}(\tau)\|^2 d\tau + \int_{[T]}^T \|\mathcal{A}^{-l}z_{tt}(\tau)\|^2 d\tau \\ &\leq C_R \sum_{k=0}^{[T]} (\|z_t(k)\|_{-1}^2 + \|\nabla z(k)\|^2), \end{aligned}$$

where $[T]$ denotes the integer part of T . Now we can apply the stabilizability estimate in (68) with $t = k$ for each k and obtain (72). \square

Proof of Theorem 3.11

We use the idea due to Málek–Nečas [29] (see also [30] and [13]).

For some $T \geq 1$ which we specify latter and for some $l > \max\{d, 6\}/4$ we consider the space

$$W_T = \left\{ u \in C(0, T; H_0^1(\Omega)) : u_t \in C(0, T; H^{-1}(\Omega)), u_{tt} \in L_2(0, T; H^{-2l}(\Omega)) \right\}$$

with the norm

$$|u|_{W_T}^2 = \max_{t \in [0, T]} [\|\nabla u(t)\|^2 + \|u_t(t)\|_{-1}^2] + \int_0^T \|u_{tt}(t)\|_{-2l}^2 dt.$$

Let \mathfrak{A}_T be the set of weak solutions to (1) on the interval $[0, T]$ with initial data $(u(0); u_t(0))$ from the attractor \mathfrak{A} . It is clear that \mathfrak{A}_T is a closed bounded set in W_T . Indeed, if the sequence of solutions $u^n(t)$ with initial data in \mathfrak{A}_T is fundamental in W_T , then we have that $u^n(0) \rightarrow u_0$ strongly in $H^1(\Omega)$, $u^n(0) \rightarrow u_0$ weakly in $L_{p+1}(\Omega)$ and $u_t^n(0) \rightarrow u_1$ weakly in $L_2(\Omega)$ for some $(u_0; u_1) \in \mathfrak{A}$. By (13) and (72) this implies that $u^n(t)$ converges in W_T to the solution with initial data $(u_0; u_1)$. This yields the closeness of \mathfrak{A}_T in W_T . The boundedness of \mathfrak{A}_T is obvious.

On \mathfrak{A}_T we define the shift operator V by the formula

$$V : \mathfrak{A}_T \mapsto \mathfrak{A}_T, \quad [Vu](t) = u(T + t), \quad t \in [0, T].$$

It is clear that \mathfrak{A}_T is strictly invariant with respect to V , i.e. $V\mathfrak{A}_T = \mathfrak{A}_T$. It follows from (13) and (72) that

$$|VU_1 - VU_2|_{W_T} \leq C_T |U_1 - U_2|_{W_T}, \quad U_1, U_2 \in \tilde{\mathcal{B}}_T.$$

By Proposition 3.14 we have that

$$\begin{aligned} \max_{s \in [0, T]} \{ \|z_t(T+s)\|_{-1}^2 + \|\nabla z(T+s)\|^2 \} &\leq ae^{-\gamma T} \max_{s \in [0, T]} \{ \|z_t(s)\|_{-1}^2 + \|\nabla z(s)\|^2 \} \\ &\quad + b \int_0^{2T} \left[\|z(\tau)\|^2 + \|\mathcal{A}^{-l} z_t(\tau)\|^2 \right] d\tau, \end{aligned}$$

where $a, b, \gamma > 0$ depends on the size of the set \mathfrak{A} in $\mathcal{H}_1 = [H^2 \cap H_0^1](\Omega) \times H_0^1(\Omega)$. Lemma 3.15 and Proposition 3.14 also yield that

$$\int_T^{2T} \|\mathcal{A}^{-l} z_{tt}\|^2 d\tau \leq Ce^{-\gamma T} (\|z_t(0)\|_{-1}^2 + \|\nabla z(0)\|^2) + C(1+T) \int_0^{2T} \left[\|z\|^2 + \|\mathcal{A}^{-l} z_t\|^2 \right] d\tau.$$

Therefore we obtain that

$$|VU_1 - VU_2|_{W_T}^2 \leq q_T |U_1 - U_2|_{W_T}^2 + C_T [n_T^2(U_1 - U_2) + n_T^2(VU_1 - VU_2)] \quad (73)$$

for every $U_1, U_2 \in \mathfrak{A}_T$, where $q_T = Ce^{-\gamma T}$ and the seminorm $n_T(U)$ has the form

$$n_T^2(U) \equiv \int_0^T \left[\|u\|^2 + \|\mathcal{A}^{-l} u_t\|^2 \right] d\tau \quad \text{for } U = \{u(t)\} \in W_T.$$

One can see that this seminorm is compact on W_T . Therefore we can choose $T \geq 1$ such that $q_T < 1$ in (73) and apply Theorem 2.15[13] to conclude that \mathfrak{A}_T has a finite fractal dimension in W_T . One can also see that $\mathfrak{A} = \{(u(t); u_t(t))_{t=s} : u(\cdot) \in \mathfrak{A}_T\}$ does not depend on s . Therefore the fractal dimension of \mathfrak{A} is finite in the space $\tilde{\mathcal{H}} = H_0^1(\Omega) \times H^{-1}(\Omega)$. By interpolation argument it follows from (66) and (13) that $S(t)|_{\mathfrak{A}}$ is a Hölder continuous mapping from $\tilde{\mathcal{H}}$ into \mathcal{H}_r for each $t > 0$. Since $\dim_f^{\tilde{\mathcal{H}}} \mathfrak{A} < \infty$, this implies that $\dim_f^{\mathcal{H}_r} \mathfrak{A}$ is finite.

3.5 Attractor in the energy space. Non-supercritical case

In this section we deal with the attractor in the strong topology of the energy space which we understand in the standard sense (see, e.g., [2, 9, 21, 44]). Namely, the *global attractor* of the evolution semigroup $S(t)$ is defined as a bounded closed set $\mathfrak{A} \subset \mathcal{H}$ which is strictly invariant ($S(t)\mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and uniformly attracts all other bounded sets:

$$\lim_{t \rightarrow \infty} \sup \{ \text{dist}_{\mathcal{H}}(S(t)y, \mathfrak{A}) : y \in B \} = 0 \quad \text{for any bounded set } B \text{ in } \mathcal{H}.$$

Since $\mathcal{H} = H_0^1(\Omega) \times L_2(\Omega)$ in the non-supercritical case, Theorem 3.9 implies the existence of a compact set in \mathcal{H} which attracts bounded sets in the strong topology. This leads to the following assertion.

Theorem 3.16 *Let Assumptions 1.1 and 3.4 be in force. Assume also that $\phi(s)$ is strictly positive (i.e., $\phi(s) > 0$ for all $s \in \mathbb{R}_+$) and $f'(s) \geq -c$ for all $s \in \mathbb{R}$ in the non-supercritical case (when the bounds in (6) are not valid). Then the evolution semigroup $S(t)$ possesses a compact global attractor \mathfrak{A} in \mathcal{H} . This attractor \mathfrak{A} coincides with the partially strong attractor given by*

Theorem 3.11 and thus (i) $\mathfrak{A} \subset \mathcal{H}_1 = [H^2 \cap H_0^1](\Omega) \times H_0^1(\Omega)$; (ii) the relation in (66) hold; (iii) $\mathfrak{A} = \mathbb{M}_+(\mathcal{N})$, where \mathcal{N} is the set of equilibria (see (67)). Moreover, we have that

$$\text{dist}_{\mathcal{H}}(y, \mathcal{N}) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ for any } y \in \mathcal{H}. \quad (74)$$

If in addition we assume Assumption 3.12(ii), then \mathfrak{A} has a finite fractal dimension in the space $\mathcal{H}_r = [H^{1+r} \cap H_0^1](\Omega) \times H^r(\Omega)$ for every $r < 1$.

Proof. We apply Theorems 3.11 and 3.13. To obtain (74) we only note that by Remark 3.3 the semigroup $S(t)$ is gradient on the *whole* space \mathcal{H} . Thus the standard results on gradient systems (see, e.g., [2, 9, 44]) lead to the conclusion in (74). \square

Under additional hypotheses we can establish other dynamical properties of the system under the consideration. We impose now the following set of requirements.

Assumption 3.17 We assume that $\phi \in C^2(\mathbb{R}_+)$ is a nondecreasing function ($\phi'(s) \geq 0$ for $s \geq 0$), $f'(s) \geq -c$ for some $c \geq 0$, and one of the following requirements fulfills:

- (a) either f is subcritical: either $d \leq 2$ or (5) holds with $p < p_* \equiv (d+2)(d-2)^{-1}$, $d \geq 3$;
- (b) or else $3 \leq d \leq 6$, $f \in C^2(\mathbb{R})$ is critical, i.e.,

$$|f''(u)| \leq C(1 + |u|^{p_*-2}), \quad u \in \mathbb{R}, \quad p_* = (d+2)(d-2)^{-1}.$$

Our second main result in this section is the following theorem.

Theorem 3.18 Let Assumptions 1.1(ii), 3.12, and 3.17 be in force. Then

- (1) Any trajectory $\gamma = \{(u(t); u_t(t)) : t \in \mathbb{R}\}$ from the attractor \mathfrak{A} given by Theorem 3.16 possesses the properties

$$(u; u_t; u_{tt}) \in L_\infty(\mathbb{R}; [H^2 \cap H_0^1](\Omega) \times H_0^1(\Omega) \times L_2(\Omega)) \quad (75)$$

and there is $R > 0$ such that

$$\sup_{\gamma \in \mathfrak{A}} \sup_{t \in \mathbb{R}} (\|\Delta u(t)\|^2 + \|\nabla u_t(t)\|^2 + \|u_{tt}(t)\|^2) \leq R^2. \quad (76)$$

- (2) There exists a fractal exponential attractor \mathfrak{A}_{exp} in \mathcal{H} .
- (3) Let $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ be a finite set of functionals on $H_0^1(\Omega)$ and

$$\epsilon_{\mathcal{L}} = \epsilon_{\mathcal{L}}(H_0^1(\Omega), L_2(\Omega)) \equiv \sup \{ \|u\| : u \in H_0^1(\Omega), \quad l_j(u) = 0, j = 1, \dots, N, \|u\|_1 \leq 1 \}$$

be the corresponding completeness defect.; Then there exists $\varepsilon_0 > 0$ such that under the condition $\epsilon_{\mathcal{L}} \leq \varepsilon_0$ the set \mathcal{L} is (asymptotically) determining in the sense that the property

$$\lim_{t \rightarrow \infty} \max_j \int_t^{t+1} |l_j(u^1(s) - u^2(s))|^2 ds = 0$$

implies that $\lim_{t \rightarrow \infty} \|S(t)y_1 - S(t)y_2\|_{\mathcal{H}} = 0$. Here above $S(t)y_i = (u^i(t); \partial_t u^i(t))$, $i = 1, 2$.

We recall (see, e.g., [16] and also [9, 13, 14]) that a compact set $\mathfrak{A}_{\text{exp}} \subset \mathcal{H}$ is said to be a fractal exponential attractor for the dynamical system $(\mathcal{H}, S(t))$ iff $\mathfrak{A}_{\text{exp}}$ is a positively invariant set of finite fractal dimension in \mathcal{H} and for every bounded set $D \subset \mathcal{H}$ there exist positive constants t_D , C_D and γ_D such that

$$d_X\{S(t)D \mid \mathfrak{A}_{\text{exp}}\} \equiv \sup_{x \in D} \text{dist}_{\mathcal{H}}(S(t)x, \mathfrak{A}_{\text{exp}}) \leq C_D \cdot e^{-\gamma_D(t-t_D)}, \quad t \geq t_D.$$

We also mentioned that the notion of determining functionals goes back to the papers by Foias and Prodi [18] and by Ladyzhenskaya [25] for the 2D Navier-Stokes equations. For the further development of the theory we refer to [15] and to the survey [8], see also the references quoted in these publications. We note that for the first time determining functionals for second order (in time) evolution equations with a nonlinear damping was considered in [10], see also a discussion in [14, Section 8.9]. We also refer to [8] and [9, Chap.5] for a description of sets of functionals with small completeness defect.

Proof of Theorem 3.16

The main ingredient of the proof is some quasi-stability property of $S(t)$ in the energy space \mathcal{H} which is stated in the following assertion.

Proposition 3.19 (Strong quasi-stability) *Suppose that Assumptions 1.1(ii), 3.12 and 3.17 hold. Let $u^1(t)$ and $u^2(t)$ be two weak solutions such that $\|(u^i(0); u_t^i(0))\|_{\mathcal{H}} \leq R$, $i = 1, 2$, then their difference $z(t) = u^1(t) - u^2(t)$ satisfies the relation*

$$\|z_t(t)\|^2 + \|\nabla z(t)\|^2 \leq a_R (\|z_t(0)\|^2 + \|\nabla z(0)\|^2) e^{-\gamma_R t} + b_R \int_0^t e^{-\gamma_R(t-\tau)} \|z(\tau)\|^2 d\tau, \quad (77)$$

where a_R, b_R, γ_R are positive constants.

Proof. As a starting point we consider the energy type relation (51) for the difference z (which we already use in the proof of the second part of Proposition 3.2) and estimate the term

$$G(t) \equiv (G(u^1, u^2; t), z_t) = H_1(t) + H_2(t) + H_3(t)$$

given by (52) using the additional hypotheses imposed. One can see that

$$|H_1(t)| \leq \varepsilon \|\nabla z_t\|^2 + C_{R,\varepsilon} (\|\nabla u_t^1\|^2 + \|\nabla u_t^2\|^2) \|\nabla z\|^2.$$

Here and below we use the fact that $\|u_t^i\|^2 + \|\nabla u^i(t)\|^2 \leq C_R$ for all $t \geq 0$ (see (48)).

We also have that

$$H_2(t) = \frac{1}{2} \frac{d}{dt} \left[\tilde{\phi}_{12}(t) |(\nabla(u^1 + u^2), \nabla z)|^2 \right] + \hat{H}_2(t),$$

where $|\hat{H}_2(t)| \leq C_R (\|\nabla u_t^1\| + \|\nabla u_t^2\|) \|\nabla z\|^2$.

If f is subcritical, i.e., Assumption 3.17(a) holds, then the estimate for $H_3(t)$ is direct:

$$|H_3(t)| \leq C_R \|\nabla z_t\| \|z\|_{1-\delta} \leq \varepsilon (\|\nabla z_t\|^2 + \|\nabla z\|^2) + C_{R,\varepsilon} \|z\|^2$$

for some $\delta > 0$ and for any $\varepsilon > 0$. Therefore in the argument below we concentrate on the critical case described in Assumption 3.17(b). In this case we have that

$$H_3(t) = \frac{1}{2} \frac{d}{dt} \left[\int_0^1 \int_{\Omega} f'(u^2 + \lambda(u^1 - u^2)) |z|^2 d\lambda dx \right] + \hat{H}_3(t),$$

where

$$\hat{H}_3(t) = -\frac{1}{2} \int_0^1 \int_{\Omega} f''(u^2 + \lambda(u^1 - u^2)) (u_t^2 + \lambda(u_t^1 - u_t^2)) |z|^2 d\lambda dx.$$

By the growth condition of f'' we have that

$$|\hat{H}_3(t)| \leq C \int_{\Omega} [1 + |u^1|^{p_*-2} + |u^2|^{p_*-2}] (|u_t^1| + |u_t^2|) |z|^2 dx.$$

Therefore the Hölder inequality and the Sobolev embedding $H^1(\Omega) \subset L_{p_*+1}(\Omega)$ imply that

$$\begin{aligned} |\hat{H}_3(t)| &\leq C \left[1 + \|u^1\|_{L_{p_*+1}(\Omega)}^{p_*-2} + \|u^2\|_{L_{p_*+1}(\Omega)}^{p_*-2} \right] [\|u_t^1\|_{L_{p_*+1}(\Omega)} + \|u_t^2\|_{L_{p_*+1}(\Omega)}] \|z\|_{L_{p_*+1}(\Omega)}^2 \\ &\leq C_R [\|\nabla u_t^1\| + \|\nabla u_t^2\|] \|\nabla z\|^2. \end{aligned}$$

Now we introduce the energy type functional

$$\begin{aligned} E_*(t) &= \frac{1}{2} \|z_t\|^2 + \frac{1}{4} \phi_{12}(t) \|\nabla z\|^2 \\ &\quad + \frac{1}{2} \left[\int_0^1 \int_{\Omega} f'(u^2 + \lambda(u^1 - u^2)) |z|^2 d\lambda dx + \tilde{\phi}_{12}(t) |(\nabla(u^1 + u^2), \nabla z)|^2 \right]. \end{aligned}$$

From (51) and the calculations above we obviously have that

$$\frac{d}{dt} E_*(t) + \left[\frac{1}{2} \sigma_{12}(t) - \varepsilon \right] \|\nabla z_t\|^2 \leq C_{R,\varepsilon} \left[d_{12}(t) + \sqrt{d_{12}(t)} \right] \|\nabla z\|^2,$$

where $d_{12}(t) = \|\nabla u_t^1(t)\|^2 + \|\nabla u_t^2(t)\|^2$. Therefore using Lemma 2.3 we obtain that the function

$$W_*(t) = E_*(t) + \eta \left[(z, z_t) + \frac{1}{4} \sigma_{12}(t) \|\nabla z\|^2 \right], \quad \eta > 0,$$

satisfies the relation

$$\begin{aligned} \frac{d}{dt} W_*(t) + \left[\frac{1}{2} \sigma_{12}(t) - \varepsilon \right] \|\nabla z_t\|^2 - \eta \|z_t\|^2 + \eta \left[\frac{1}{2} \phi_{12}(t) \|\nabla z\|^2 + \tilde{\phi}_{12}(t) |(\nabla(u^1 + u^2), \nabla z)|^2 \right] \\ + \eta \int_0^1 \int_{\Omega} f'(u^2 + \lambda(u^1 - u^2)) |z|^2 d\lambda dx \leq \varepsilon \|\nabla z\|^2 + C_{R,\varepsilon} d_{12}(t) \|\nabla z\|^2. \end{aligned}$$

Therefore, if we introduce $\tilde{W}(t) = W_*(t) + C \|z(t)\|^2$ with appropriate $C > 0$ and with $\eta > 0$ small enough, then we obtain that

$$a_R (\|z_t(t)\|^2 + \|\nabla z(t)\|^2) \leq \tilde{W}(t) \leq b_R (\|z_t(t)\|^2 + \|\nabla z(t)\|^2)$$

and

$$\frac{d}{dt}\tilde{W}(t) + c_R\tilde{W}(t) \leq C_R d_{12}(t)\|\nabla z\|^2 + C\|z(t)\|^2$$

with positive constants. Thus the finiteness of the integral in (48) and the standard Gronwall's argument implies the result in (77) in the critical case. In the subcritical case we use the same argument but for the functional E_* without the term containing f' . \square

Completion of the proof of Theorem 3.16: Proposition 3.19 means that the semigroup $S(t)$ is quasi-stable on the absorbing set \mathcal{B}_0 defined in Remark 3.6 in the sense of Definition 7.9.2 [14]. Therefore to obtain the result on regularity stated in (75) and (76) we first apply Theorem 7.9.8 [14] which gives us that

$$\sup_{t \in \mathbb{R}} (\|\nabla u_t(t)\|^2 + \|u_{tt}(t)\|^2) \leq C_{\mathfrak{A}} \quad \text{for any trajectory } \gamma = \{(u(t); u_t(t)) : t \in \mathbb{R}\} \subset \mathfrak{A}.$$

Applying (66) we obtain (75) and (76).

By (11) any weak solution $u(t)$ possesses the property

$$\int_t^{t+1} \|u_{tt}(\tau)\|^2 d\tau \leq C_{R,T} \quad \text{for } t \in [0, T], \quad \forall T > 0,$$

provided $(u_0; u_1) \in S(1)\mathcal{B}_0$, where \mathcal{B}_0 is the absorbing set defined in Remark 3.6. This implies that $t \mapsto S(t)y$ is a $1/2$ -Hölder continuous function with values in \mathcal{H} for every $y \in S(1)\mathcal{B}_0$. Therefore the existence of a fractal exponential attractor follows from Theorem 7.9.9 [14].

To prove the statement concerning determining functionals we use the same idea as in the proof of Theorem 8.9.3 [14].

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